# OPTIMAL CONSUMPTION FROM INVESTMENT AND RANDOM ENDOWMENT IN INCOMPLETE SEMIMARTINGALE MARKETS

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Abstract. We consider the problem of maximizing expected utility from consumption in a constrained incomplete semimartingale market with a random endowment process, and establish a general existence and uniqueness result using techniques from convex duality. The notion of asymptotic elasticity of Kramkov and Schachermayer is extended to the time-dependent case. By imposing no smoothness requirements on the utility function in the temporal argument, we can treat both pure consumption and combined consumption/terminal wealth problems, in a common framework. To make the duality approach possible, we provide a detailed characterization of the enlarged dual domain which is reminiscent of the enlargement of $\mathbb{L}^1$ to its topological bidual $(\mathbb{L}^\infty)^*$, a space of finitely-additive measures. As an application, we treat the case of a constrained Itô-process market-model.

## 1. Introduction

Both modern and classical theories of economic behavior use utility functions to describe the amount of "satisfaction" of financial agents depending on their wealth or consumption rate. Starting with an initial endowment, an agent is faced with the problem of distributing wealth among financial assets with different degrees of uncertainty. If the market is arbitrage-free, the agent can never "beat the market", but may still invest in such a way as to maximize expected utility. A considerable body of literature has been devoted to this subject. First to consider the utility maximization problem in continuous-time stochastic financial market models was Merton in [Mer69], [Mer71]. He used a strong assumption (usually not justified in practice) that stock-prices are governed by Markovian dynamics with constant coëfficients. In this way he could use the methods of stochastic programming and in particular, the Bellman-Hamilton-Jacobi equation of dynamic programming. More recently, a "martingale" approach to the problem in complete Itô-process markets was introduced by Pliska [Pli86], Karatzas, Lehoczky and Shreve [KLS87] and Cox and Huang [CH89], [CH91]. They related the marginal utility from the terminal wealth of the optimal portfolio to the density of the (unique) martingale measure, using powerful convex-duality techniques. Difficulties with this approach arise in incomplete markets. The main idea here is to use the convex nature of the problem, to formulate and solve a dual variational problem, and then proceed as in the complete case. In discrete-time and on a finite probability space, the problem was studied by He and Pearson [HP91a], and in a continuous-time model by of G.-L. Xu in his doctoral dissertation [Xu90], by He and Pearson [HP91b] and by Karatzas, Lehoczky, Shreve and Xu [KLSX91]. In the paper [KS99], Kramkov and

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Schachermayer solve the problem in the context of a general incomplete semimartingale financial market. They show that a necessary and sufficient condition for the existence of an optimal solution is *reasonable asymptotic elasticity* of the utility function. This is an analytic condition on the behavior of the utility function at infinity, which excludes certain pathological situations. These authors also show that the set of densities of local martingale measures is too small to host the solutions of the dual problem. Thus, they enlarge it to a suitably chosen set $\mathcal{Y}$ of supermartingales, in a manner reminiscent of enlarging $L^1$ to its topological bidual $(L^\infty)^*$. Although these supermartingales cannot be used directly as pricing rules for derivative securities, Kramkov and Schachermayer show this is possible under an appropriate change of numéraire.

When, in addition to initial wealth, the agent faces an uncertain random intertemporal endowment, the situation becomes technically much more demanding and the gap between complete and incomplete markets even more apparent. In the complete market setting, the entire uncertain endowment can be "hedged away" in the market, and the problem becomes equivalent to the one where the entire endowment process is replaced by its present value, in the form of an augmented initial wealth. A self-contained treatment of this situation, in Itô-process models for financial markets can be found in Section 4.4 of the monograph by Karatzas and Shreve [KS98]. An otherwise complete market with random endowment, where the incompleteness is introduced through prohibition of borrowing against future income, is dealt with in [KJP98]. In incomplete markets, several authors consider this problem in various degrees of generality. We mention Cuoco who deals with a cone-constrained Itô-process market with random endowment in [Cuo97] - he attacks directly the primal problem circumventing the duality approach altogether, at the cost of rather strict restrictions on the utility function. A definitive solution to the problem of maximizing of utility from terminal wealth in incomplete (though not constrained in a more general way) semimartingale markets with random endowment is offered in [CSW01]. The main contribution of that paper is the introduction of finitely-additive measures into the realm of optimal stochastic control problems encountered in mathematical finance. The essential difference between utility maximization with and without random endowment is probably best described by the authors of [CSW01]:

> *" it was not important in the analysis of* [KS99] *where the 'singular mass of* $\hat{\mathbb{Q}}$ *has disappeared to'. In the present paper this becomes very important . . . [it] acts on the accumulated random endowment and can be located in* $(\mathbb{L}^\infty)^*$*".*

We finally mention [Sch00] as an extensive survey of the optimal investment theory.

This paper strives to complement the existing results in several ways. First, we incorporate intertemporal consumption in the optimization problem. We are dealing with an agent investing in an incomplete market, where prices are modelled by an arbitrary semimartingale with right-continuous and left-limited paths. From the present moment to some finite time horizon $T$, our agent is not only deciding how to manage a portfolio by dynamically readjusting the positions in various financial assets, but also choosing a portion of wealth to be consumed and not further reinvested. The agent also has to take into account the uncertainty in the random endowment stream. It is from this consumption, or from consumption and terminal wealth, that utility is derived. We allow the utility function to be random, reflecting the changes in agent's risk-preferences from one time to another.

In a departure from existing theory, we do not impose any smoothness on the utility function in its temporal argument. As a result, we have a common framework for problems that involve consumption only *and* for problems that involve both consumption and terminal wealth. In addition to dealing with an inherently incomplete semimartingale market-model, we impose convex cone constraints on the investment choices the agent is facing. In this way we can model incompleteness and prohibition of short-sales, to name only two.

For utility functions we formulate the concept of asymptotic elasticity and, under an appropriate condition of "reasonable asymptotic elasticity", we establish existence and uniqueness of the optimal consumption-investment strategy. In [KS99] it was only the terminal value of a dual process that appeared in the analysis, the dual domain $\{Y_T : Y \in \mathcal{Y}\} \subseteq L^0_+$ being endowed with the topology of convergence in probability. The more difficult situation in [CSW01] required the dual domain to be extended to the closure of the set of all equivalent martingale measures in $(\mathbb{L}^\infty)^*$ - a space whose elements are *finitely-additive set-functions*. Abusing terminology slightly, we shall call such set-functions "finitelly-additive measures". In our case, we have to mimic the natural correspondence between measures and uniformly integrable martingales in the finitely-additive world. It turns out that the right choice consists of a dual domain, inhabited by finitely-additive measures, and coupled with supermartingales corresponding to the Radon-Nikodym derivatives of their regular parts. We prove rigorously that these supermartingales essentially correspond to the supermartingales in the set $\mathcal{Y}$ defined in [KS99]. The basic tool in this endeavor is the Filtered Bipolar Theorem of [Žit00].

As applications of our results, we treat two special cases - a constrained Itô-process market, where we prove that the optimal dual process is always a local martingale, and the "totally incomplete" case of Lakner and Slud ([LS91]), where the agent is not allowed to invest in the stock-market at all.

We should stress that one main motivation behind this work is the rôle it plays as a necessary step for an offensive on the problem of existence and uniqueness for equilibrium in continuous-time incomplete markets with random endowments, a task we plan to attempt in future research.

The part of our analysis dealing with duality, and especially the structure of the proof of the main result, is closely based on and inspired by the expositions in [KS99] and [CSW01]. In Section 2 we set up the market-model, and present a characterization of admissible consumption strategies. Section 3 displays our main result and Appendix A its proof. In Section 4 we give an application of our results through two examples.

## 2. The model

2.1. **The financial market.** We introduce a model for a financial market consisting of

(i) a positive, adapted process $B = (B_t)_{t\in[0,T]}$ with paths that are RCLL (Right-Continuous on $[0, T)$, with Left-Limits everywhere on $(0, T]$) and uniformly bounded from above and away from zero. We interpret $B$ as the numéraire asset - a bond, for example.

(ii) a RCLL-semimartingale $S = (S_t)_{t\in[0,T]}$ taking values in $\mathbb{R}^d$; its component processes represent the prices of $d$ risky assets, discounted in terms of the numéraire $B$.

All processes are defined on a stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with a finite **time horizon** $T > 0$, and the filtration $\mathbb{F} \triangleq (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions; $\mathcal{F}_0$ is the completion of the trivial $\sigma$-algebra.

We concentrate our attention on a financial agent endowed with initial wealth $x > 0$ and a random **cumulative endowment process** $\mathcal{E} = (\mathcal{E}_t)_{t\in[0,T]}$ - in that the total (cumulative) amount of endowment received by time $t$ is $\mathcal{E}_t$. We assume that $\mathcal{E}_0 = 0$ and $\mathcal{E}$ is nondecreasing, $\mathbb{F}$-adapted, RCLL and uniformly bounded from above, i.e., $\mathcal{E}_T \in \mathbb{L}^\infty_+(\mathbb{P})$. Similarly to the price-process $S$, we assume that $\mathcal{E}$ is already discounted (denominated in terms of $B$).

Faced with inherent uncertainty in future endowment, the agent dynamically adjusts positions in different financial assets and designates a part of wealth for immediate consumption, in the following manner:

(a) the agent chooses an $S$-integrable and $\mathbb{F}$-predictable process $H$ taking values in $\mathbb{R}^d$. The process $H$ has a natural interpretation as **portfolio process**; in other words, the $i^{\text{th}}$ component of $H_t$ is the number of shares of stock $i$ held at time $t$.

To exclude pathologies such as doubling schemes, we choose to impose the condition of **admissibility** on the agent's choice of portfolio process $H$, by requiring that the **gains process** $\int_0^\cdot H_u\, dS_u$ be uniformly bounded from below by some constant (for the theory of stochastic integration with respect to RCLL semimartingales, and the related notion of integrability, the reader may consult [Pro90]). Moreover, we ask our agent to obey the investment restrictions imposed on the structure of the market, by choosing the portfolio process $H$ in a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^d$. The set $\mathcal{K}$ represents constraints on portfolio choice, and can be used to model, for example, short-sale constraints or unavailability of some stocks for investment.

(b) apart from the choice of portfolio process, the agent chooses a nonnegative, nondecreasing $\mathbb{F}$-adapted RCLL process $C = (C_t)_{t\in[0,T]}$. The **cumulative consumption process** $C$ represents the total amount (just like $S$ and $\mathcal{E}$, already discounted by $B$) spent on consumption, up to and including time $t$.

A pair $(H, C)$ that satisfies (a) and (b) above, is called an **investment-consumption strategy**. The wealth of an agent that employs the investment-consumption strategy $(H, C)$ is given by

$$W^{H,C}_t \triangleq x + \mathcal{E}_t + \int_0^t H_u\, dS_u - C_t, \quad 0 \le t \le T. \tag{2.1}$$

If the strategy $(H, C)$ is such that the corresponding wealth process $W^{H,C}$ satisfies $W^{H,C}_T \ge 0$ a.s., we say that $(H, C)$ is an **admissible strategy**. If, for a consumption process $C$, we can find a portfolio process $H$ such that $(H, C)$ is admissible, we call $C$ an **admissible consumption process**, and say that $C$ can be **financed** by $x + \mathcal{E}$ and $H$. Let $\mu$ be an **admissible measure**, i.e., a probability measure on $[0, T]$, diffuse on $[0, T)$, such that $\mu([0, t]) < 1$ for all $t < T$. For such a measure we define the **support** $\operatorname{supp}\mu$ to be $[0, T]$ if $\mu$ charges $\{T\}$, and $[0, T)$ otherwise.

We shall be mostly interested in admissible consumption processes $C$ that can be expressed as

$$C_t = \int_0^t c(u)\, \mu(\mathrm{d}u), \quad 0 \le t \le T.$$

The set of all densities $c(\cdot)$ of such processes will be denoted by $\mathcal{A}^{\mu}(x+\mathcal{E})$. We allow for bulk consumption at the terminal time in order to be able to deal later on with utility from the terminal wealth and/or from consumption, in the same framework.

*Remark* 1. Even though we allow debt to incur before time $T$, the agent must invest in such a way as to be able to post a non-negative wealth by the end of the trading horizon, with certainty. Furthermore, the boundedness of the process $\mathcal{E}=(\mathcal{E}_t)_{t\in[0,T]}$ guarantees that the negative part of the wealth will remain bounded by a constant (a weak form of "constrained borrowing").

The following notation will be used repeatedly in the sequel:

$$\mathcal{X} \triangleq \Big\{x+\int_0^{\cdot} H_u\,dS_u\ :\ H \text{ is predictable and } S\text{-integrable, } H_t\in\mathcal{K} \text{ a.s.}$$
$$\text{for every } t\in[0,T],\ x\geq 0, \text{ and } x+\int_0^{\cdot} H_u\,dS_u \text{ is nonnegative}\Big\}, \tag{2.2}$$

2.2. **The optimization problem.** Let us introduce now a preliminary version of the optimization problem, and lay out an outline of its solution. The goal is to find a consumption-density process $\hat{c}^x(\cdot)$, financed by the initial wealth $x$ and the random endowment $\mathcal{E}$, which maximizes the expected utility from consumption - the average felicity of an agent who follows the consumption strategy $\hat{c}^x(\cdot)$. The expected utility from a consumption density process $c(\cdot)$ is given by

$$\mathbb{E}\left[\int_0^T U(t,c(t))\,\mu(dt)\right],$$

where $U$ denotes a (random) utility function and $\mu$ a utility measure. We postpone discussion of the definition and regularity properties of $U$ until Section 3. In this notation,

$$\hat{c}^x = \mathrm{argmax}_{c\in\mathcal{A}^{\mu}(x+\mathcal{E})}\,\mathbb{E}\left[\int_0^T U(t,c(t))\,\mu(dt)\right]. \tag{2.3}$$

As it is customary in the duality approach to stochastic optimization, we introduce a problem dual to (2.3) by setting

$$Y^{\hat{\mathbb{Q}}^y} = \mathrm{argmin}_{\mathbb{Q}\in\mathcal{D}}\left[\mathbb{E}\int_0^T V(t,Y_t^{\mathbb{Q}})\,dt + y\langle \mathbb{Q},\mathcal{E}_T\rangle\right].$$

Here $\mathcal{D}$ denotes the domain for the dual problem; it is the closure of the set of all supermartingale measures for the stock process $S$. The process $Y^{\mathbb{Q}}$ is a supermartingale version of the density process of $\mathbb{Q}$, and $V$ is the convex conjugate of $U$.

In the following subsections, we introduce and describe the dual domain $\mathcal{D}$ in detail, and establish some of its properties - the prominent one being weak * compactness. It is precisely this compactness property that will ensure the existence of a solution to the dual problem and - through standard tools of convex duality - the existence of an optimal consumption process $\hat{c}^x$ for any positive initial wealth $x$.

2.3. **Connections with Stochastic Control Theory.** The portfolio process $H$ serves as the analogue of the control-process in Stochastic Control Theory. It is important, though, to stress that we are not dealing here with a *partially (incompletely) observed* problem (a terminology borrowed again from Control Theory). Incomplete markets in Mathematical Finance correspond to a setting, in which the controller has full information about many aspects of the system (the market), but

various exogenously imposed constraints (taxation, transaction costs, bad credit rating, legislature, etc.) prevent him/her from choosing the control (portfolio) outside a given constraint set. In fact, even without government-imposed portfolio constraints, financial markets will typically not offer tradeable assets corresponding to a variety of sources of uncertainty (weather conditions, non-listed companies, etc.) The financial agent will still observe many of these sources, as their uncertainty evolves, but will typically not be able to "trade in all of them", as it were.

This fundamental nature of financial markets is reflected in our modelling: in Sections 1, 2 and 3, we allow the filtration $\mathbb{F}$ (with respect to which the controls are adapted) to be possibly larger that the filtration generated by the stock-price process $S$. The only requirement we impose, in the next subsection, is the one of *absence of arbitrage*, the fulfilment of which depends heavily on the choice of filtration $\mathbb{F}$. To sum up, the *observables* in financial modelling constitute a much larger class than the mere stocks we are allowed to invest in. With such an understanding, our portfolios *are* adapted only to the observables of the system. Such a setting corresponds to the well-established control-theoretic notion of admitting "open loop" controls in our analysis.

In the more specialized setup of Section 4, the filtration $\mathbb{F}$ is taken as the augmentation of the filtration generated by the Brownian motions driving the stock-prices, assuming as we do in the beginning of Subsection 4.1 that the volatility matrix process $\sigma(t)$ is *non-singular* a.s., for each $t$. At the level of generality considered in the paper, the filtration corresponding to the stock prices will be smaller than the filtration generated by the Brownian motion. But the two filtrations *are* actually the same, when interest-rates, volatilities and appreciation-rates are functions of past-and-present stock prices; this includes the case of Markovian or deterministic coefficients. In this case, "open loop" and "closed loop" (i.e., $S$-adapted) controls, actually coincide.

Finally, we would like to stress that market incompleteness is the main source of technical and conceptual problems we had to overcome in this work, whereas the case of complete markets has been well studied by many authors before; see, for instance, Chapters 3 and 4 in [KS98]. All of our results concerning the structure of the dual domain (as well as the introduction of the dual domain in the first place) are consequences of the incompleteness of the market. We are actually allowing for *two* separate sources of incompleteness - the general structure of the stock-prices, as well as the portfolio constraints in the form of the cone $\mathcal{K}$. By choosing $\mathcal{K} = \mathbb{R}^n \times \{0\} \times \cdots \times \{0\}$ for some $n = 1, \cdots d-1$, we capture exactly the setting of an incomplete market with $n$ stocks, and with $d > n$ sources of randomness that affect the coefficients in the model.

2.4. **Absence of arbitrage, finitely-additive set-functions, and the dual domain.** In order to make possible a meaningful mathematical treatment of the optimization problem, we outlaw arbitrage opportunities by postulating the existence of an **equivalent supermartingale measure**, i.e., a probability measure on $(\Omega, \mathcal{F})$, equivalent to $\mathbb{P}$, under which the elements of the set $\mathcal{X}$ in (2.2) become supermartingales. The set of all equivalent supermartingale probability measures will be denoted by $\mathcal{M}$, and we shall assume throughout that $\mathcal{M} \neq \emptyset$. A detailed treatment of the connections between various notions of arbitrage and the existence of equivalent martingale (local martingale,

supermartingale) measures, culminating with the Fundamental Theorem of Asset Pricing, can be found in [DS93] and [DS98].

As was pointed out in [CSW01], the duality treatment of utility maximization requires a nontrivial enlargement of $\mathcal{M}$: this space turns out to be too small, in terms of closedness and compactness properties. Accordingly, we define $\mathcal{D}$ to be the $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-closure of $\mathcal{M}$ in $(\mathbb{L}^\infty)^*$ – the topological dual of $\mathbb{L}^\infty$ – where $\mathcal{M}$ is canonically identified with its embedding into $(\mathbb{L}^\infty)^*$. We shall denote by $(\mathbb{L}^\infty)^*_+$ the set of non-negative elements in $(\mathbb{L}^\infty)^*$. In the following proposition we collect some properties of $(\mathbb{L}^\infty)^*$, $(\mathbb{L}^\infty)^*_+$, and $\mathcal{D}$; more information about $(\mathbb{L}^\infty)^*$ can be found in [BB83].

**Proposition 2.1.** (i) *The space $(\mathbb{L}^\infty)^*$ consists of finite, finitely-additive measures on $\mathcal{F}$, which assign the value zero to $\mathbb{P}$-null subsets of $\mathcal{F}$.*

(ii) *Under the canonical pairing $\langle\ ,\ \rangle : (\mathbb{L}^\infty)^* \times \mathbb{L}^\infty \to \mathbb{R}$, the relation $\langle \mathbb{Q}, 1\rangle = 1$ holds for all $\mathbb{Q} \in \mathcal{D}$. In other words, with the notation $\mathbb{Q}(A) \triangleq \langle \mathbb{Q}, 1_A\rangle$ for $A \in \mathcal{F}$ and $\mathbb{Q} \in (\mathbb{L}^\infty)^*$, we have $\mathbb{Q}(\Omega) = 1$ for all $\mathbb{Q} \in \mathcal{D}$.*

(iii) *$\mathcal{D}$ is weak * (i.e., $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$) – compact.*

(iv) *Every element $\mathbb{Q}$ of $(\mathbb{L}^\infty)^*_+$ admits a unique decomposition*

$$\mathbb{Q} = \mathbb{Q}^r + \mathbb{Q}^s, \quad \textit{with} \quad \mathbb{Q}^r, \mathbb{Q}^s \in (\mathbb{L}^\infty)^*_+ ,$$

*where the* **regular part** *$\mathbb{Q}^r$ is the maximal* countably-additive *measure on $\mathcal{F}$ dominated by $\mathbb{Q}$, and the* **singular part** *$\mathbb{Q}^s$ is* purely finitely-additive*, i.e., does not dominate any nontrivial countably-additive measure.*

(v) *$\mathbb{Q} \in (\mathbb{L}^\infty)^*_+$ is singular, if and only if for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{F}$ such that $\mathbb{P}(A_\varepsilon) > 1 - \varepsilon$ and $\mathbb{Q}(A_\varepsilon) = 0$.*

(vi) *Suppose a bounded sequence $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ in $(\mathbb{L}^\infty)^*_+$ is such that $\frac{d\mathbb{Q}^r_n}{d\mathbb{P}} \to f$ a.s., for some $f \geq 0$. Then any weak * cluster point $\mathbb{Q}$ of $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ satisfies $\frac{d\mathbb{Q}^r}{d\mathbb{P}} = f$ a.s. where $\mathbb{Q}^r$ denotes the regular part of $\mathbb{Q}$.*

(vii) *The regular-part operator $\mathbb{Q} \mapsto \mathbb{Q}^r$ is additive on $(\mathbb{L}^\infty)^*_+$.*

*Proof.*

(i) See [BB83], Corollary 4.7.11.

(ii) Follows from density of $\mathcal{M}$ in $\mathcal{D}$.

(iii) This is the content of Alaoglu's theorem (see [Woj96], Theorem 2.A.9).

(iv) See Theorem 10.2.1 in [BB83].

(v) See Lemma A.1. in [CSW01].

(vi) See Proposition A.1. in [CSW01].

(vii) Let $\mathbb{Q}$ and $\mathcal{R}$ be elements of $(\mathbb{L}^\infty)^*_+$. It is clear that $\mathbb{Q}^r + \mathcal{R}^r$ is a countably additive measure dominated by $\mathbb{Q} + \mathcal{R}$, so $(\mathbb{Q} + \mathcal{R})^r \geq \mathbb{Q}^r + \mathcal{R}^r$. For the equality, it is enough to show that $(\mathbb{Q} + \mathcal{R}) - (\mathbb{Q}^r + \mathcal{R}^r) = \mathbb{Q}^s + \mathcal{R}^s$ is singular. For any $\varepsilon > 0$, by (v), we can find sets $A_\varepsilon$ and $B_\varepsilon$ such that $P(A_\varepsilon) > 1 - \frac{\varepsilon}{2}$, $P(B_\varepsilon) > 1 - \frac{\varepsilon}{2}$ and $\mathbb{Q}^s(A_\varepsilon) = \mathcal{R}^s(B_\varepsilon) = 0$. With $C_\varepsilon \triangleq A_\varepsilon \cap B_\varepsilon$ we have $P(C_\varepsilon) > 1 - \varepsilon$ and $(\mathbb{Q}^s + \mathcal{R}^s)(C_\varepsilon) = 0$; this completes the proof, by appeal to (v).

□

*Remark* 2. In the light of property (ii) we may interpret the elements of $\mathcal{D}$ as finitely-additive probability measures on $\mathcal{F}$, weakly absolutely continuous with respect to $\mathbb{P}$.

For our analysis, it will be necessary to associate a nonnegative RCLL supermartingale $Y^{\mathbb{Q}} = (Y_t^{\mathbb{Q}})_{t\in[0,T]}$ to every $\mathbb{Q}\in\mathcal{D}$. For $\mathbb{Q}\in\mathcal{M}$, this process is just the RCLL-modification of the martingale $(\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t])_{t\in[0,T]}$. For general $\mathbb{Q}\in(\mathbb{L}^\infty)^*_+$, the construction of $Y^{\mathbb{Q}}$ is rather delicate (cf. (2.5) below). To make headway on this issue, we let $\mathbb{Q}^r$ denote the regular part of $\mathbb{Q}$ and, for any $\sigma$-algebra $\mathcal{G}\subseteq\mathcal{F}$, we denote by $\mathbb{Q}|_{\mathcal{G}}$ the restriction of the set-function $\mathbb{Q}$ to $\mathcal{G}$. Since the regular-part operator $\mathbb{Q}\mapsto\mathbb{Q}^r$ depends nontrivially on the domain of $\mathbb{Q}$, we stress that $(\mathbb{Q}|_{\mathcal{G}})^r$ stands for a countably-additive measure on $\mathcal{G}$ and, in general, does *not* equal $\mathbb{Q}^r|_{\mathcal{G}}:$ the regular-part and restriction operations do not commute, in general. In fact, we have the following result:

**Proposition 2.2.** *For any two $\sigma$-algebras $\mathcal{G}\subseteq\mathcal{H}$ and every $\mathbb{Q}\in(\mathbb{L}^\infty)^*$, we have $(\mathbb{Q}|_{\mathcal{G}})^r\geq(\mathbb{Q}|_{\mathcal{H}})^r|_{\mathcal{G}}$.*

*Proof.* By definition, $(\mathbb{Q}|_{\mathcal{G}})^r$ is the maximal countably-additive measure on $\mathcal{G}$ dominated by $\mathbb{Q}$, so it must dominate $(\mathbb{Q}|_{\mathcal{H}})^r|_{\mathcal{G}}$ – another countably-additive measure on $\mathcal{G}$ dominated by $\mathbb{Q}$. □

For $\mathbb{Q}\in\mathcal{D}$ we define the process

$$L_t^{\mathbb{Q}} \triangleq \frac{d(\mathbb{Q}|_{\mathcal{F}_t})^r}{d(\mathbb{P}|_{\mathcal{F}_t})},\quad t\in[0,T]. \tag{2.4}$$

It is exactly the property from Proposition 2.2 that makes then the process defined by

$$Y_t^{\mathbb{Q}} \triangleq \liminf_{q\searrow t,\ q \text{ is rational}} L_q^{\mathbb{Q}},\quad 0\leq t<T,\quad\text{and}\quad Y_T^{\mathbb{Q}}\triangleq L_T^{\mathbb{Q}} \tag{2.5}$$

a RCLL supermartingale. This, seemingly unnatural, regularization through the limit-inferior in (2.5) is necessary, since there is no guarantee that an RCLL-modification exists for the process $L^{\mathbb{Q}}$. Appendix I, theorem 4, p. 395 and Theorem 10, p. 402 in [DM82] establish good measurability properties of the processes involved, as well as the fact that the limit-inferior in (2.5) is actually a true limit for every $t\in[0,T)$, on a subset of $\Omega$ of full probability. When $\mathbb{Q}\in\mathcal{M}$, it is immediate that the process $Y^{\mathbb{Q}}=(Y_t^{\mathbb{Q}})_{t\in[0,T]}$ of (2.5) is the RCLL-modification of the martingale $(\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t])_{t\in[0,T]}$.

We define the two sets of processes

$$\mathcal{Y}^{\mathcal{M}}\triangleq\left\{Y^{\mathbb{Q}}\,:\,\mathbb{Q}\in\mathcal{M}\right\}\quad\text{and}\quad\mathcal{Y}^{\mathcal{D}}\triangleq\left\{Y^{\mathbb{Q}}\,:\,\mathbb{Q}\in\mathcal{D}\right\}\supseteqq\mathcal{Y}^{\mathcal{M}}. \tag{2.6}$$

The following proposition goes deeper into the properties of the elements of $\mathcal{Y}^{\mathcal{D}}$. It shows that the regularization in the definition (2.5) of the process $Y^{\mathbb{Q}}$ is, in fact, a harmless operation.

**Proposition 2.3.** (a) *For $\mathbb{Q}\in\mathcal{D}$, there exists a countable set $K\subseteq[0,T)$, such that $Y_t^{\mathbb{Q}}=L_t^{\mathbb{Q}}$ for all $t\in[0,T]\setminus K$, almost surely. In particular, $Y^{\mathbb{Q}}=L^{\mathbb{Q}}$ $(\mu\otimes\mathbb{P})$-a.e., for any admissible measure $\mu$.*

(b) *For every stopping time $S$, we have $Y_S^{\mathbb{Q}}\leq\frac{d(\mathbb{Q}|_{\mathcal{F}_S})^r}{d(\mathbb{P}|_{\mathcal{F}_S})}$ a.s.*

*Proof.* (a) Let $K$ be the set of discontinuity points of the decreasing function $t\mapsto\mathbb{E}[L_t^{\mathbb{Q}}]=(\mathbb{Q}|_{\mathcal{F}_t})^r(\Omega)$, on $[0,T)$; this set is at most countable. For every $t<T$, Fatou's lemma gives

$$Y_t^{\mathbb{Q}}\leq\liminf_{q\searrow t,\ \text{q is rational}}\mathbb{E}[L_q^{\mathbb{Q}}|\mathcal{F}_t]\leq L_t^{\mathbb{Q}}. \tag{2.7}$$

On the other hand, for any sequence of rationals $\{q_n\}_{n\in\mathbb{N}}$ with $q_n \searrow t$, $\{L^{\mathbb{Q}}_{q_n}\}_{n\in\mathbb{N}}$ is a backward supermartingale bounded in $\mathbb{L}^1$, so that $L^{\mathbb{Q}}_{q_n} \to Y^{\mathbb{Q}}_t$ both in $\mathbb{L}^1$ and a.s., thanks to the Backward Supermartingale Convergence Theorem (see [Chu74], Theorem 9.4.7, page 338). For each $t \in [0,T]\setminus K$ we have thus $\mathbb{E}[Y^{\mathbb{Q}}_t] = \mathbb{E}[L^{\mathbb{Q}}_t]$ which, together with (2.7) and the fact that $K$ is at most countable, completes the proof of (a).

(b) For an arbitrary stopping time $S$, and $n \in \mathbb{N}$, we put $S^n = (2^{-n}\lfloor 2^n S + 1\rfloor)\wedge T$, so that $S \le S^n \le S + 2^{-n}$. Therefore, $\{S^n\}_{n\in\mathbb{N}}$ is a sequence of stopping times with finite range, a.s. decreasing to $S$. By the definition (2.5) of $Y^{\mathbb{Q}}$ we have $Y^{\mathbb{Q}}_S = \liminf_n L^{\mathbb{Q}}_{S^n}$. Let $\{t^n_1, \dots, t^n_{m_n}\}$ be the range of $S^n$. Then for $A \in \mathcal{F}_S \subseteq \mathcal{F}_{S_n}$ we have

$$\begin{aligned}\mathbb{E}[Y^{\mathbb{Q}}_S \mathbf{1}_A] &= \mathbb{E}[\liminf_n \hat{Y}^{\mathbb{Q}}_{S^n}\cdot \mathbf{1}_A] \le \liminf_n \mathbb{E}[\hat{Y}^{\mathbb{Q}}_{S^n}\cdot \mathbf{1}_A] = \liminf_n \sum_{k=1}^{m_n} \mathbb{E}[\hat{Y}^{\mathbb{Q}}_{t^n_k}\cdot \mathbf{1}_{A\cap\{S^n = t^n_k\}}] \\ &\le \liminf_n \sum_{k=1}^{m_n} \langle \mathbb{Q}, \mathbf{1}_{A\cap\{S^n=t^n_k\}}\rangle = \langle \mathbb{Q}, \mathbf{1}_A\rangle.\end{aligned}$$

Therefore, $Y^{\mathbb{Q}}_S$ is the density of a (countably-additive) measure dominated by $\mathbb{Q}$ on $\mathcal{F}_S$, and we conclude that $Y^{\mathbb{Q}}_S \le \frac{d(\mathbb{Q}|_{\mathcal{F}_S})^r}{d(\mathbb{P}|_{\mathcal{F}_S})}$, almost surely. □

The next results, useful for the duality treatment and interesting in their own right, introduce the notion of *Fatou-convergence*, and relate it to the more familiar notion of weak * convergence. Fatou-convergence is analogous to a.s. convergence in the context of RCLL-processes, and was used for example in [Kra96], [FK97] and [DS99].

**Definition 2.4.** Let $\{Y^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of nonnegative, $\mathbb{F}$-adapted processes with RCLL paths. We say that $\{Y^{(n)}\}_{n\in\mathbb{N}}$ **Fatou-converges** to an $\mathbb{F}$-adapted process $Y$ with RCLL-paths, if there is a countable, dense subset $\mathcal{T}$ of $[0,T]$, such that

$$Y_t = \liminf_{s\downarrow t, s\in\mathcal{T}}\Big(\liminf_n Y^{(n)}_s\Big) = \limsup_{s\downarrow t, s\in\mathcal{T}}\Big(\limsup_n Y^{(n)}_s\Big), \quad \forall\, t\in[0,T] \tag{2.8}$$

holds almost surely; we interpret (2.8) to mean $Y_t = \lim_n Y^{(n)}_t$ a.s. for $t = T$. A set of nonnegative RCLL-supermartingales is called **Fatou-closed**, if it is closed with respect to Fatou-convergence.

Before stating the next proposition we need a technical result - see Lemma 8 in [Žit00].

**Lemma 2.5.** *Let $\{Y^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of nonnegative RCLL-supermartingales, Fatou-converging to a nonnegative RCLL-supermartingale $Y$. There is a countable set $K \subseteq [0,T)$ such that $Y_t = \liminf_n Y^{(n)}_t$ for all $t\in[0,T]\setminus K$, almost surely.*

**Proposition 2.6.** *Let $\mu$ be a probability measure on $[0,T]$, diffuse on $[0,T)$. Let $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ with a cluster point $\mathbb{Q}^* \in \mathcal{D}$, such that the sequence $\{Y^{\mathbb{Q}^{(n)}}\}_{n\in\mathbb{N}}$ converges, both $(\mu\otimes\mathbb{P})$-a.e. and in the Fatou sense. Then the Fatou-limit $Y$ coincides with the $(\mu\otimes\mathbb{P})$-limit, up to a.e. equivalence, and both are equal to $Y^{\mathbb{Q}^*}$.*

*Proof.* The two limits are the same $(\mu \otimes \mathbb{P})$-a.e., by Lemma 2.5. By Proposition 2.3, there exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of countable subsets of $[0,T)$, and a $\mu$-null set $K'$, such that

$$Y_t = \lim_n Y_t^{\mathbb{Q}^{(n)}} = \lim_n L_t^{\mathbb{Q}^{(n)}}, \quad \text{for all } t \in [0,T] \setminus K$$

holds almost surely, where $K \triangleq K' \cup \bigcup_{n\in\mathbb{N}} K_n$. By Proposition 2.1(vi), (2.4), and Proposition 2.3, there is a $\mu$-null set $\hat{K} \supseteq K$ such that

$$Y_t = Y_t^{\mathbb{Q}^*} = L_t^{\mathbb{Q}^*}, \text{ for all } t \in [0,T] \setminus \hat{K}$$

holds almost surely. Since $[0,T] \setminus \hat{K}$ is dense in $[0,T]$, the right-continuous processes $Y$ and $Y^{\mathbb{Q}^*}$ are indistinguishable. □

2.5. **On a point raised by Cvitanić, Schachermayer and Wang.** In [KS99], page 6, the authors define a set $\mathcal{Y}$ of supermartingales, which acts as an enlargement for the set of densities of equivalent martingale measures; they then use $\mathcal{Y}$ as the domain for the convex-duality approach to utility maximization in incomplete markets. In their setup there is no endowment after time $t = 0$, no portfolio constraint, and utility comes from terminal wealth only. In terms of the set $\mathcal{X}$ of stochastic integrals in (2.2), the set $\mathcal{Y}$ of supermartingales is defined as

$$\begin{aligned} \mathcal{Y} \quad \triangleq \quad & \Big\{ Y : Y \text{ is an adapted nonnegative RCLL process such that } Y_0 \le 1 \text{ and} \\ & \big(Y_t X_t\big)_{t\in[0,T]} \text{ is a supermartingale for each process } X \in \mathcal{X} \Big\}. \end{aligned} \tag{2.9}$$

Obviously, the elements of $\mathcal{Y}$ are supermartingales (just take $H = 0$, thus $X \equiv x$, in (2.2)), and $\mathcal{Y}$ contains the set $\mathcal{Y}^{\mathcal{M}}$ of (2.6) by its very definition; but except in trivial cases, $\mathcal{Y}$ is a true enlargement of $\mathcal{Y}^{\mathcal{M}}$. An attempt to study the structure of $\mathcal{Y}$ was made in [Žit00], by establishing and applying a generalization of the *Bipolar Theorem for Subsets of* $\mathbb{L}^0_+$ (see [BS99]); this is a non-locally-convex version of the classical Bipolar Theorem of functional analysis. The aforementioned generalization comes in the form of the *Filtered Bipolar Theorem*, whose statement and relevant definitions we recall now from [Žit00]:

**Definition 2.7.** A set of $\mathcal{Y}$ of nonnegative $\mathbb{F}$-adapted processes with RCLL paths, is said to be

(1) **(process-) solid**, if for each $Y \in \mathcal{Y}$ and each nonincreasing $\mathbb{F}$-adapted process $B$ with RCLL paths and $B_0 \le 1$, we have $YB \in \mathcal{Y}$;
(2) **fork-convex**, if for any $s \in (0,T]$, any $h \in L^0_+(\mathcal{F}_s)$ with $h \le 1$ a.s., and any $Y^{(1)}, Y^{(2)}, Y^{(3)} \in \mathcal{Y}$, the process $Y$ defined by

$$Y_t = \left\{ \begin{array}{cc} Y_t^{(1)} & , \ 0 \le t < s \\ Y_s^{(1)} \left( h \frac{Y_t^{(2)}}{Y_s^{(2)}} + (1-h) \frac{Y_t^{(3)}}{Y_s^{(3)}} \right) & , \ s \le t \le T \end{array} \right\}$$

belongs to $\mathcal{Y}$.

**Definition 2.8.** Let $\mathcal{Y}$ be a set of nonnegative, $\mathbb{F}$-adapted with RCLL paths. The **(process)-polar** of $\mathcal{Y}$ is the set of all nonnegative, $\mathbb{F}$-adapted processes $X$ with RCLL paths, such that $XY = (X_tY_t)_{t\in[0,T]}$ is a supermartingale with $(XY)_0 \le 1$ for all $Y \in \mathcal{Y}$.

We can now state a mild extension of the main result of [Žit00]. The additional statement (last sentence of Theorem 2.9 below) follows directly from the proof of the original version.

**Theorem 2.9.** **[Filtered Bipolar Theorem]** *Let $\mathcal{Y}$ be a set of nonnegative, $\mathbb{F}$-adapted processes with RCLL paths, $Y_0 \leq 1$ for each $Y \in \mathcal{Y}$, and with $Y_T > 0$ a.s. for at least one $Y \in \mathcal{Y}$. The process-bipolar $\mathcal{Y}^{\times\times} = (\mathcal{Y}^{\times})^{\times}$ of $\mathcal{Y}$ is the smallest Fatou-closed, fork-convex and solid set of $\mathbb{F}$-adapted processes $Y$ with RCLL paths and $Y_0 \leq 1$ that contains $\mathcal{Y}$. Furthermore, every element of $\mathcal{Y}^{\times\times}$ can be obtained as the Fatou-limit of a sequence in the solid and convex hull of $\mathcal{Y}$.*

*Remark* 3. The set $\mathcal{Y}^{\mathcal{M}}$ of (2.6) is fork-convex, and its process-bipolar is the set $\mathcal{Y}$ of (2.9) (see Theorem 4 in [Žit00]). It follows immediately from Theorem 2.9 that $\mathcal{Y}$ is the solid and Fatou-closed hull of $\mathcal{Y}^{\mathcal{M}}$.

The task we take on in this subsection is to formulate and establish formally the statement put forth by the authors in [CSW01], to the effect that

> *...the idea of passing from $\mathcal{M}$ to $\mathcal{D}$* (introduced in [CSW01]) *had already been implicitly present in* [KS99] (disguised in the definition of $\mathcal{Y}$).

Namely, we shall show that $\mathcal{Y}^{\mathcal{D}} \subseteq \mathcal{Y}$, and that $\mathcal{Y}^{\mathcal{D}}$ already contains all maximal elements of $\mathcal{Y}$. More precisely, we have the following result.

**Theorem 2.10.** *The set $\mathcal{Y}^{\mathcal{D}}$ in (2.6) is fork-convex and Fatou-closed, and the set $\mathcal{Y}$ of (2.9) is its solid hull.*

*Proof.* Since $\mathcal{Y}$ is the process-bipolar of $\mathcal{Y}^{\mathcal{M}}$ from (2.6), by the Filtered Bipolar Theorem 2.9 it is enough to prove that $\mathcal{Y}^{\mathcal{D}}$ is fork-convex, contained in $\mathcal{Y}$, and Fatou-closed, since $\mathcal{Y}^{\mathcal{M}}$ is already contained in $\mathcal{Y}^{\mathcal{D}}$.

The fork-convexity of $\mathcal{Y}^{\mathcal{D}}$ follows from its definition, from the fork-convexity of $\mathcal{Y}^{\mathcal{M}}$, and from the fact (Theorem 2.9) that every $Y \in \mathcal{Y}^{\mathcal{D}} \subseteq \mathcal{Y}$ can be Fatou-approximated by a sequence in $\mathcal{Y}^{\mathcal{M}}$.

As for Fatou-closedness, we take a sequence $\{Y^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{Y}^{\mathcal{D}}$, Fatou-converging towards a supermartingale $Y$. Let $\lambda$ stand for the normalized Lebesgue measure on $[0, T]$. By Komlós's theorem (see [Sch86]) and the convexity of $\mathcal{Y}^{\mathcal{D}}$, we can assume that $\{Y^{(n)}\}_{n\in\mathbb{N}}$ converges $(\lambda \otimes \mathbb{P})$−a.e., by passing to a sequence of convex combinations if necessary (note that this operation preserves the Fatou-limit). Let $\{\mathbb{Q}^n\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be a sequence such that $Y^{(n)} = Y^{\mathbb{Q}^n}$. By the weak * compactness of $\mathcal{D}$, the sequence $\{\mathbb{Q}^n\}_{n\in\mathbb{N}}$ possesses a cluster point $\mathbb{Q}^*$. Proposition 2.6 now yields $Y = Y^{\mathbb{Q}^*}$, implying Fatou-closedness of $\mathcal{Y}^{\mathcal{D}}$.

Finally, we prove that $\mathcal{Y}^{\mathcal{D}} \subseteq \mathcal{Y}$. Let $X \in \mathcal{X}$ be such that $X_0 = 1$, and let $Y \in \mathcal{Y}^{\mathcal{D}}$. By the definition of $\mathcal{Y}$, it will be enough to show that $YX$ is a supermartingale, and by Proposition 2.3 it is enough to prove that $L^{\mathbb{Q}}X$ is a supermartingale, where $L^{\mathbb{Q}}$ is the process defined in (2.4). Equivalently, we have to prove $\langle (\mathbb{Q}|_{\mathcal{F}_s})^r, X_s \mathbf{1}_A\rangle \geq \langle (\mathbb{Q}|_{\mathcal{F}_t})^r, X_t \mathbf{1}_A\rangle$, for all $0 \leq s < t \leq T$, $A \in \mathcal{F}_s$. For this, we may assume without loss of generality that $X_s$ is bounded on $A$.

Recall that, for $\mathbb{Q} \in \mathcal{M}$, the process $X$ is a nonnegative $\mathbb{Q}$-supermartingale. By density of $\mathcal{M}$ in $\mathcal{D}$, we easily conclude that $\langle \mathbb{Q}, X_s \mathbf{1}_A\rangle \geq \langle \mathbb{Q}, (X_t \wedge m)\, \mathbf{1}_A\rangle$, for all $\mathbb{Q} \in \mathcal{D}$ and $m \in (0, \infty)$. The

regular-part-operator is positive, so we have

$$\langle (\mathbb{Q}|_{\mathcal{F}_s})^r, X_s \mathbf{1}_A\rangle + \langle (\mathbb{Q}|_{\mathcal{F}_s})^s, X_s \mathbf{1}_A\rangle \geq \langle (\mathbb{Q}|_{\mathcal{F}_t})^r, (X_t \wedge m)\, \mathbf{1}_A\rangle, \quad \forall\, m \in (0,\infty).$$

Proposition 2.1 (v) guarantees the existence of a sequence of sets $\{A_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}_s$ such that $\mathbb{P}[A_n] > 1-2^{-n}$ and $(\mathbb{Q}|_{\mathcal{F}_s})^s(A_n) = 0$. We get

$$\langle (\mathbb{Q}|_{\mathcal{F}_s})^r, X_s \mathbf{1}_{A\cap A_n}\rangle \geq \langle (\mathbb{Q}|_{\mathcal{F}_t})^r, (X_t \wedge m)\, \mathbf{1}_{A\cap A_n}\rangle,\ \forall\, m \in (0,\infty),\ n\in\mathbb{N},$$

and the claim follows by letting $m, n \to \infty$. □

For future use, we restate the result of the Theorem 2.10 in the following terms.

**Corollary 2.11.** *Every $Y \in \mathcal{Y}$ can be written as $Y = Y^{\mathbb{Q}} D$, where $\mathbb{Q} \in \mathcal{D}$, and $D$ is a nonincreasing, nonnegative, $\mathbb{F}$-adapted process with $D_0 \leq 1$ and RCLL paths. The process $Y^{\mathbb{Q}}$ can be obtained as the Fatou-limit of a sequence of martingales in $\mathcal{Y}^{\mathcal{M}}$.*

2.6. **A Characterization Of Admissible Consumption Processes.** The enlargement of the dual domain from $\mathcal{M}$ to $\mathcal{D}$ necessitates a reformulation of certain old results in the new setting. As given in subsection 2.1, the definition of an admissible consumption process is as intuitively graspable as practically useless. To remedy this situation, we establish a budget-constraint-characterization of admissible consumption processes, analogous to Theorem 3.6, p. 166 in [KS98], in the context of the endowment process $x + \mathcal{E} = (x+\mathcal{E}_t)_{t\in[0,T]}$.

**Proposition 2.12.** *A nonnegative, nondecreasing, right-continuous and $\mathbb{F}$-adapted process $C$ is an admissible cumulative consumption process, if and only if*

$$\mathbb{E}\left[\int_0^T Y_t^{\mathbb{Q}}\, dC_t\right] \leq x + \langle \mathbb{Q}, \mathcal{E}_T\rangle, \quad \textit{for all}\ \ \mathbb{Q}\in\mathcal{D}. \tag{2.10}$$

*Proof.* Let $C$ be a nonnegative nondecreasing adapted right-continuous process satisfying (2.10). For each probability measure $\mathbb{Q}\in\mathcal{M}$, the process $Y^{\mathbb{Q}}$ is the RCLL modification of the martingale $\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]$, $t\in[0,T]$. By virtue of the left-continuity and existence of right-limits for the process $t\mapsto C_{t-}$, the stochastic integral $M_t \triangleq \int_0^t C_{u-}\, dY_u^{\mathbb{Q}}$, $0\leq t\leq T$ is a local martingale ([Pro90], Theorem III. 17), so we can find a non-decreasing sequence of stopping times $\{T_n\}_{n\in\mathbb{N}}$ such that $\mathbb{P}[T_n = T]\to 1$ as $n\to\infty$, and the processes $M_\cdot^{T_n} \equiv M_{\cdot\wedge T_n}$ are uniformly integrable martingales, for each $n\in\mathbb{N}$. By the assumption (2.10) and the integration-by-parts formula, we have

$$\begin{aligned} x + \langle \mathbb{Q}, \mathcal{E}_T\rangle \quad &\geq \quad \mathbb{E}\int_0^T Y_t^{\mathbb{Q}}\, dC_t = \lim_n\ \mathbb{E}\int_0^{T_n} Y_t^{\mathbb{Q}}\, dC_t = \lim_n \left(\mathbb{E}\int_0^{T_n} Y_{t-}^{\mathbb{Q}}\, dC_t + \sum_{s\leq T_n} \Delta Y_s^{\mathbb{Q}} \Delta C_s\right) \\ &= \quad \lim_n \left(\mathbb{E}\left[Y_{T_n}^{\mathbb{Q}} C_{T_n} - \int_0^{T_n} C_{t-}\, dY_t^{\mathbb{Q}}\right]\right) = \lim_n\ \mathbb{E}_{\mathbb{Q}}[C_{T_n}] = \langle \mathbb{Q}, C_T\rangle. \end{aligned} \tag{2.11}$$

Let us define

$$Z_t \triangleq \mathrm{esssup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[C_T - \mathcal{E}_T|\mathcal{F}_t],\ 0\leq t\leq T.$$

From Theorem 2.1.1 in [KQ95], the process $Z$ is a supermartingale under *each* $\mathbb{Q}\in\mathcal{M}$, with a RCLL modification. Choose this RCLL version for $Z$. Moreover, $Z$ is uniformly bounded from below and $Z_0\leq x$; this is because $\mathbb{E}_{\mathbb{Q}}[C_T - \mathcal{E}_T]\leq x$ for every $\mathbb{Q}\in\mathcal{M}$, thanks to (2.11). Applying

the Constrained Version of the Optional Decomposition Theorem (see [FK97], Theorem 4.1) to $Z$, we can assert the existence of an admissible portfolio $\hat{H}$ and of a nondecreasing optional process $F$ with $F_0 \geq 0$, such that $Z_t = \hat{X}_t - F_t$, where $\hat{X}_t \triangleq x + \int_0^t \hat{H}_u \, dS_u$. On the other hand, by the increase of $C$ we have

$$\hat{X}_t - F_t = Z_t \geq C_t - \operatorname{essinf}_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\mathcal{E}_T | \mathcal{F}_t], \qquad t \in [0,T],$$

so that $\hat{X}_T - C_T + \mathcal{E}_T \geq F_T \geq F_0 \geq 0$, a.s., implying the admissibility of the strategy $(\hat{H}, C)$.

Conversely, let $C$ be an admissible consumption process; there exists then an admissible porfolio process $H$, such that the process $X_\cdot \triangleq x + \int_0^\cdot H_u \, dS_u$ satisfies $X_T - C_T + \mathcal{E}_T \geq 0$. By the supermartingale property of $X$ under every $\mathbb{Q} \in \mathcal{M}$, we conclude that $\langle \mathbb{Q}, C_T \rangle \leq x + \langle \mathbb{Q}, \mathcal{E}_T \rangle$, $\forall\, \mathbb{Q} \in \mathcal{M}$. Suppose first that $C$ is uniformly bounded from above by a constant $M$, and define its right-continuous inverse (taking values in $[0,\infty]$) by

$$D_s = \inf \{ t \geq 0 \, : \, C_t > s \}, \quad 0 \leq s < \infty.$$

For an arbitrary, but fixed $\mathbb{Q} \in \mathcal{D}$, by Theorem 55 in [DM82] and Fubini's theorem, we can write

$$\mathbb{E}\left[ \int_0^T Y_t^{\mathbb{Q}} \, dC_t \right] = \mathbb{E}\left[ \int_0^M Y_{D_s}^{\mathbb{Q}} \mathbf{1}_{\{D_s < \infty\}} \, ds \right] = \int_0^M \phi(s) \, ds,$$

where $\phi(s) = \mathbb{E}[Y_{D_s}^{\mathbb{Q}} \mathbf{1}_{\{D_s < \infty\}}]$. By the supermartingale property of $Y^{\mathbb{Q}}$ and the increase of $D$, the function $\phi$ is nonincreasing, so we can find a countable set $K$, dense in $[0,M]$, that contains all discontinuity points of $\phi$. For a denumeration $\{s_k\}_{k \in \mathbb{N}}$ of $K$, the topology on $\mathcal{D}$ induced by the pseudometric

$$d(\mathbb{Q}_1, \mathbb{Q}_2) = |\langle \mathbb{Q}_1 - \mathbb{Q}_2, C_T \rangle| + \sum_k 2^{-n} |\langle \mathbb{Q}_1 - \mathbb{Q}_2, \mathbf{1}_{\{D_{s_k} < \infty\}} \rangle|$$

is coarser than the weak * topology on $\mathcal{D}$, so we can find a *sequence* $\{\mathbb{Q}^n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ such that

$$\langle \mathbb{Q}^n, C_T \rangle \to \langle \mathbb{Q}, C_T \rangle \quad \text{and} \quad \langle \mathbb{Q}^n, \mathbf{1}_{\{D_s < \infty\}} \rangle \to \langle \mathbb{Q}, \mathbf{1}_{\{D_s < \infty\}} \rangle, \quad \text{as } n \to \infty,$$

for every $s \in K$. Such choice for the sequence $\{\mathbb{Q}^n\}_{n \in \mathbb{N}}$ implies that $\phi^n(s) = \mathbb{E}_{\mathbb{Q}^n}[\mathbf{1}_{\{D_s < \infty\}}]$ converges to $\langle \mathbb{Q}, \mathbf{1}_{\{D_s < \infty\}} \rangle$ for *every* $s$. Using again Theorem 55 in [DM82], the integration-by-parts formula from the first part of the proof, and the Dominated Convergence Theorem, we get

$$\begin{aligned} x + \langle \mathbb{Q}, \mathcal{E}_T \rangle \; &= \; x + \lim_n \langle \mathbb{Q}^n, \mathcal{E}_T \rangle \; \geq \; \lim_n \langle \mathbb{Q}^n, C_T \rangle \; = \; \lim_n \mathbb{E}\left[ \int_0^T Y_t^{\mathbb{Q}^n} \, dC_t \right] = \lim_n \int_0^M \phi^n(s) \\ &= \; \int_0^M \langle \mathbb{Q}, \mathbf{1}_{\{D_s < \infty\}} \rangle \, ds. \end{aligned}$$

As $D_s$ is a stopping time, Proposition 2.3,(b) yields

$$\int_0^M \langle \mathbb{Q}, \mathbf{1}_{\{D_s < \infty\}} \rangle \, ds \; \geq \; \int_0^M \mathbb{E}\left[ \frac{d(\mathbb{Q}|_{\mathcal{F}_{D_s}})^r}{d(\mathbb{P}|_{\mathcal{F}_{D_s}})} \mathbf{1}_{\{D_s < \infty\}} \right] ds \; \geq \; \int_0^M \mathbb{E}[Y_{D_s}^{\mathbb{Q}} \mathbf{1}_{\{D_s < \infty\}}] \, ds \; = \; \mathbb{E} \int_0^T Y_t^{\mathbb{Q}} \, dC_t,$$

which establishes the claim.

We turn now to the case of $C$ which is not necessarily bounded. For $M \in \mathbb{N}$, the truncated consumption process $C^M = C \wedge M$ is admissible and (2.10) holds with $C$ replaced by $C \wedge M$. Passing to the limit as $M \to \infty$ on the left-hand side of (2.10) is justified by the increase of the trajectories of $C$ and the Monotone Convergence Theorem. □

*Remark* 4. The necessity for the rather lengthy and technical proof of this result (to be more precise: the authors' inability to find a shorter one), stems from two rather unpleasant facts: first, $(\mathbb{L}^\infty)^*$ is not metrizable, and secondly, Fubini's theorem fails in the setting of finitely-additive measures (see [YH52], Theorem 3.3, p. 57 for such a counterexample).

## 3. The optimization problem

3.1. **The Preference Structure.** Apart from external factors, such as market conditions and the randomness of the endowment process $\mathcal{E}$, it is important to describe the agent's "preference structure" (or idiosyncratic rapport with risk). We shall adopt the von Neyman-Morgenstern utility approach to risk-aversion, and proceed to define a utility random field $U : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}$.

We shall impose no smoothness conditions in the time parameter. Instead, we shall control the range of the marginal utility. As seen in [KS99], a condition of *reasonable asymptotic elasticity* (in the setting of an incomplete semimartingale market with initial endowment only, and utility from terminal wealth) is both necessary and sufficient for the existence of an optimal investment policy. This is the reason for extending the notion of asymptotic elasticity to the time-dependent case, and for restricting our analysis to reasonably elastic utilities only. More precisely, we have the following definition.

**Definition 3.1.** A jointly measurable function $U : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ is called a **(reasonably elastic) utility random field**, if it has the following properties (unless specified otherwise, all these properties are assumed to hold almost surely and the argument $\omega \in \Omega$ will consistently be suppressed):

(1) For a fixed $t \in [0,T]$, $U(t,\cdot)$ is strictly concave, increasing and $C^1$ satisfying the so-called Inada conditions $\partial_2 U(t,0+) = \infty$ and $\partial_2 U(t,\infty) = 0$. In other words, $U(t,\cdot)$ is a **utility function**.

(2) There are continuous, strictly decreasing (nonradom) functions $K_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $K_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \in [0,T]$ and $x > 0$, we have $K_1(x) \leq \partial_2 U(t,x) \leq K_2(x)$, and $\limsup_{x\to\infty} \frac{K_2(x)}{K_1(x)} < \infty$.

(3) At $x = 1$, $t \mapsto U(t,1)$ is a uniformly bounded function of $(\omega, t)$ and $\lim_{x\to\infty}(\operatorname{essinf}_{t,\omega} U(t,x)) > 0$.

(4) $U$ is **reasonably elastic**, i.e., its asymptotic elasticity satisfies $\mathrm{A}E[U] < 1$ a.s., where

$$\mathrm{A}E[U] := \limsup_{x\to\infty} \left( \operatorname{esssup}_{t,\omega} \frac{x\partial_2 U(t,x)}{U(t,x)} \right).$$

(5) For any $x > 0$, the stochastic process $U(\cdot, x)$ is $\mathbb{F}$-progressively measurable.

*Remark* 5. Condition 3 is the least restrictive - in fact, it only serves to simplify the analysis by excluding some trivial nuisances, as well as to have the expression AE$[U]$ of part 4 well defined. It is an immediate consequence of conditions 2 and 3 that the function $t \to U(x_0, t)$ is bounded for any $x_0 > 0$, a.s. Also, the trajectory $U(t,\infty)$ is either a bounded function of $t$, or we have $U(t,\infty) = \infty$ for all $t$, a.s.

**Example 3.2.** Let $\hat{U} : \mathbb{R}_+ \to \mathbb{R}_+$ be a utility function as in Definition 3.1 (1), with $\hat{U}(\infty) > 0$ and $\limsup_{x\to\infty} \frac{x\hat{U}'(x)}{\hat{U}(x)} < 1$. Let $\psi$ be a measurable function of $[0,T]$ such that $0 < \inf_{t\in[0,T]} \psi(t) \leq \sup_{t\in[0,T]} \psi(t) < \infty$. Then it is easy to see that $U(t,x) \triangleq \psi(t)\hat{U}(x)$ is a reasonably elastic utility random field. In particular, this example includes so-called *discounted* time-dependent utility functions of the form $U(t,x) = e^{-\beta t}\hat{U}(x)$.

**Example 3.3.** Let $U_1 : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ be a deterministic utility field with corresponding $K_1$ and $K_2$ as in Definition 3.1,(2). Further, let $U_2 : \mathbb{R}_+ \to \mathbb{R}$ be a utility function satisfying

$$U_2(\infty) > 0, \ \limsup_{x\to\infty} \frac{xU_2'(x)}{U_2(x)} < 1 \quad \text{and} \quad 0 < \liminf_{x\to\infty} \frac{U_2'(x)}{K_1(x)} \leq \limsup_{x\to\infty} \frac{U_2'(x)}{K_1(x)} < \infty.$$

One can check then the requirements of Definition 3.1 to see that

$$U(t,x) := \left\{ \begin{array}{ll} U_1(t,x), & t < T \\ U_2(x), & t = T \end{array} \right\}$$

is a reasonably elastic utility random field.

**Example 3.4.** Let $U_1 : [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ be any deterministic reasonably elastic utility field, and let $B_t$ be a adapted process uniformly bounded from above and away from zero. To model a stochastic discount factor, we define $U(t,x) \triangleq U_1(t, B_t x)$. Such a utility random field arises when the agent accrues utility from nominal, instead of real value of consumption.

With a utility random field $U$ we associate a random field $V : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$V(t,y) \triangleq \sup_{x>0}[U(t,x) - xy], \quad 0 < y < \infty, \tag{3.1}$$

the **conjugate of** $U$. We also define the random field $I : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}$, by $I(t,y) = (\partial_2 U(t,\cdot))^{-1}(y)$, the **inverse marginal utility** of $U$. The following proposition lists some important, though technical, properties of these random fields and their conjugates. They will be used extensively in the sequel. We leave the proof to the diligent reader.

**Proposition 3.5.** *Let $U$ be a utility random field and $V$ its conjugate.*

(1) *There are (deterministic) utility functions $\underline{U}$ and $\overline{U}$ such that*

$$\underline{U}(x) \leq U(t,x) \leq \overline{U}(x) \text{ for all } x > 0 \text{ and all } t \in [0,T], \text{ a.s.}$$

(2) *For a given $t \in [0,T]$, the function $V(t,\cdot)$ is finite valued, strictly decreasing, strictly convex and continuously differentiable.*

(3) *The convex conjugates $\underline{V}$ and $\overline{V}$ of $\underline{U}$ and $\overline{U}$ satisfy*

$$\underline{V}(y) \leq V(t,y) \leq \overline{V}(y) \text{ for all } y > 0, \text{ and all } t \in [0,T] \text{ a.s.}$$

*In particular, the function $t \mapsto V(t,y)$ is uniformly bounded, for any $y \in (0,\infty)$.*

**Definition 3.6.** Any utility functions (i.e., strictly concave, increasing, and continuously differentiable functions that satisfy the Inada conditions) $\underline{U} : \mathbb{R}_+ \to \mathbb{R}$ and $\overline{U} : \mathbb{R}_+ \to R$, such that $\underline{U}(x) \leq U(t,x) \leq \overline{U}(x)$ for all $x > 0$ and $t \in [0,T]$, are called a ***minorant*** and a ***majorant*** of $U$, respectively. Functions $\underline{V} : \mathbb{R}_+ \to \mathbb{R}$ and $\overline{V} : \mathbb{R}_+ \to \mathbb{R}$, that are convex conjugates of some minorant and majorant of $U$, are called a ***minorant*** and a ***majorant*** of $V$, respectively.

*Remark* 6. It follows immediately from the definition of convex conjugation that for any minorant and majorant $\underline{V}$ and $\overline{V}$ of $V$, we have $\underline{V}(y) \leq V(t,y) \leq \overline{V}(y)$, for all $y > 0$ and $t \in [0,T]$.

Finally, we state a technical result stemming from the reasonable-asymptotic-elasticity condition; its proof is, mutatis mutandis, identical to the proof leading to Corollary 6.3., page 994 of [KS99].

**Proposition 3.7.** *Let $U$ be a utility random field. If we define the random sets*

$$\begin{aligned}
\Gamma_1 &= \Big\{\gamma > 0 \,:\, \exists x_0 > 0,\ \forall t \in [0,T],\ \forall \lambda > 1,\ \forall x \geq x_0,\quad U(t,\lambda x) < \lambda^\gamma U(t,x)\Big\} \\
\Gamma_2 &= \Big\{\gamma > 0 \,:\, \exists x_0 > 0,\ \forall t \in [0,T],\ \forall x \geq x_0,\quad \partial_2 U(t,x) < \gamma \frac{U(t,x)}{x}\Big\} \\
\Gamma_3 &= \Big\{\gamma > 0 \,:\, \exists y_0 > 0,\ \forall t \in [0,T],\ \forall\, 0 < \rho < 1,\ \forall\, 0 < y \leq y_0,\quad V(t,\rho y) < \rho^{-\frac{\gamma}{1-\gamma}} V(t,y)\Big\} \\
\Gamma_4 &= \Big\{\gamma > 0 \,:\, \exists y_0 > 0,\ \forall t \in [0,T],\ \forall\, 0 < y \leq y_0,\quad -\partial_2 V(t,y) < \frac{\gamma}{1-\gamma}\frac{V(t,y)}{y}\Big\},
\end{aligned}$$

*then* $\inf \Gamma_1 = \inf \Gamma_2 = \inf \Gamma_3 = \inf \Gamma_4 = \mathrm{A}E[U]$*, a.s.*

3.2. **The Optimization Problem and the Main Result.** The principal task our agent is facing, is how to control investment and consumption, in order to achieve maximal expected utility. At this point we have defined all notions necessary to cast this question in precise mathematical terms.

**Problem 3.8.** Let $U$ be a utility random field, $\mathcal{E}$ a cumulative endowment process, and $\mu$ an admissible measure on $[0,T]$ as defined in subsection 2.1. For an initial capital $x > 0$, we are to characterize the value function

$$\mathfrak{U}(x) \triangleq \sup_{c \in \mathcal{A}^\mu(x+\mathcal{E})} \mathbb{E}\left[\int_0^T U(t,c(t))\,\mu(dt)\right] \qquad \textsf{(Primal problem)}. \tag{3.2}$$

*Remark* 7. When the above $\mu \otimes \mathbb{P}-$integral fails to exist, we set its value to be $-\infty$. This is equivalent to the approach taken in [KS98] where the authors consider only consumption processes such that the negative part $U^-(t,c(t))$ is $\mu \otimes \mathbb{P}-$integrable.

To avoid trivial situations we adopt the following

**Standing Assumption 3.9.** There exists $x > 0$ such that $\mathfrak{U}(x) < \infty$.

*Remark* 8. Due to the boundedness of $\mathcal{E}_T$, the Standing Assumption 3.9 will hold under any conditions that will guarantee finiteness of the value function $\mathfrak{U}$, when $\mathcal{E}_T \equiv 0$. One such a condition is given by $0 \leq U(t,x) \leq \kappa(1+x^\alpha)$, $\forall x > 0$, $t \in [0,T]$, for some constants $\kappa > 0$ and $\alpha \in (0,1)$. For details, see Remark 3.9, p. 274 in [KS98], and compare with [KLSX91] and [Xu90].

Together with the Primal problem we set up the Dual Problem with value function

$$\mathfrak{V}(y) \triangleq \inf_{\mathbb{Q} \in \mathcal{D}} J(y,\mathbb{Q}), \text{ where } J(y,\mathbb{Q}) \triangleq \left(\mathbb{E}\left[\int_0^T V(t, yY_t^{\mathbb{Q}})\,\mu(dt)\right] + y\langle \mathbb{Q}, \mathcal{E}_T\rangle\right),\ y > 0 \quad \textsf{(Dual problem)}. \tag{3.3}$$

It will be shown below that the Dual problem is in fact well-posed, i.e., the integral in its definition always exists in $\bar{\mathbb{R}}$. The main result of this paper is then as follows:

**Theorem 3.10.** *Let $\mathcal{E} = (\mathcal{E}_t)_{t\in[0,T]}$ be a cumulative endowment process, and $\mu$ an admissible measure. Furthermore, let $U$ be a utility random field, let $V$ be its conjugate, and let $\mathfrak{U}$ and $\mathfrak{V}$ be the value functions of the Primal and the Dual Problem, respectively. Under the Standing Assumption 3.9, the following assertions hold:*

(i) $|\mathfrak{U}(x)| < \infty$ *for all* $x > 0$ *and* $|\mathfrak{V}(y)| < \infty$ *for all* $y > 0$*, i.e., the value functions are finite throughout their domain.*

(ii) *The value functions $\mathfrak{U}$ and $\mathfrak{V}$ are continuously differentiable, $\mathfrak{U}$ is strictly concave and $\mathfrak{V}$ is strictly convex.*

(iii) $\mathfrak{U}(x) = \inf_{y>0}[\mathfrak{V}(y) + xy]$*, and* $\mathfrak{V}(y) = \sup_{x>0}[\mathfrak{U}(x) - xy]$ *for* $x, y > 0$*, i.e. $\mathfrak{U}$ and $\mathfrak{V}$ are convex conjugates of each other.*

(iv) *The derivatives $\mathfrak{U}'$ and $\mathfrak{V}'$ of the value functions satisfy:*

$$\lim_{y\to 0} -\mathfrak{V}'(y) = \lim_{x\to 0} \mathfrak{U}'(x) \in [\inf_{\mathbb{Q}\in\mathcal{D}} \langle \mathbb{Q}, \mathcal{E}_T\rangle, \sup_{\mathbb{Q}\in\mathcal{D}} \langle \mathbb{Q}, \mathcal{E}_T\rangle],$$
$$\lim_{y\to\infty} \mathfrak{V}'(y) = \lim_{x\to\infty} \mathfrak{U}'(x) = 0.$$

(v) *Both Primal and Dual Problem have solutions $\hat{c}^x \in \mathcal{A}^\mu(x+\mathcal{E})$ and $\hat{\mathbb{Q}}^y \in \mathcal{D}$, respectively, for all $x, y > 0$. For $x > 0$ and $y > 0$ related by $\mathfrak{U}'(x) = y$, we have*

$$\hat{c}^x(t) = I(t, yY_t^{\hat{\mathbb{Q}}^y}), \quad 0 \le t \le T,$$

*where $\hat{\mathbb{Q}}^y$ is a solution to the Dual problem corresponding to $y$. Furthermore, $\hat{c}^x$ is the unique optimal consumption-rate process , and $\hat{\mathbb{Q}}^y$ is determined uniquely as far as the process $Y^{\hat{\mathbb{Q}}^y}$ and the action of $\hat{\mathbb{Q}}^y$ on $\mathcal{E}_T$ are concerned.*

(vi) *The derivative $\mathfrak{V}'(y)$ satisfies*

$$\mathfrak{V}'(y) = \langle \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle - \mathbb{E}\left[\int_0^T Y_t^{\hat{\mathbb{Q}}^y} I(t, yY_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right],$$

*where $\hat{\mathbb{Q}}^y$ is the solution to the Dual problem corresponding to $y$.*

**Example 3.11.** Let $U_1$ be a utility random field and $U_2$ a utility function. Consider the problem of maximizing expected utility from consumption and terminal wealth

$$\mathfrak{U}(x) := \sup\left(\mathbb{E}\left[\int_0^T U_1(t, c(t))\mathrm{d}t + U_2(X_T)\right]\right), \tag{3.4}$$

where the supremum is taken over all admissible investment-consumption strategies. This problem can be regarded as a special case of our Primal problem. Indeed, if we view the terminal wealth as being consumed instantaneously, we can translate (3.4) into

$$\mathfrak{U}(x) = \sup_{c\in\mathcal{A}^\mu(x+\mathcal{E})} \mathbb{E}\left[\int_0^T U(t, c(t))\,\mu(dt)\right]$$

where $\mu = \frac{1}{2T}\lambda + \frac{1}{2}\delta_{\{T\}}$ ($\lambda$ denotes the Lebesgue measure on $[0, T]$) and

$$U(t, x) := \left\{\begin{array}{cc} 2TU_1(t, \frac{x}{2T}) & t < T \\ 2U_2(\frac{x}{2}) & t = T \end{array}\right\},$$

if $U_1$ and $U_2$ satisfy the requirements of Example 3.3. In this case $C_T - C_{T-} = \frac{1}{2}c(T)$ plays the role of terminal wealth.

## 4. Examples

4.1. **The Itô-process model.** We specialize the specifications of our model as follows:

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic base supporting a $d$-dimensional Brownian motion $W = (W_t)_{t\in[0,T]}$, and we assume that $\mathbb{F} \triangleq (\mathcal{F}_t)_{t\in[0,T]}$ is the augmentation of the filtration generated by $W$. The market coëfficients are given by a bounded real-valued **interest rate** process $r$, a bounded **appreciation rate** process $b$ taking values in $\mathbb{R}^d$ and a $(d\times d)$-matrix valued **volatility** process $\sigma$. We assume that $r$, $b$ and $\sigma$ are progressively measurable and $\sigma(t)$ is a symmetric regular matrix for each $t$, with all eigenvalues uniformly bounded from above and away from zero, almost surely.

The dynamics of the money market (numeraire asset) and the stock market is given by

$$\left\{ \begin{array}{rcll} dB_t &=& B_t r(t)dt, & B_0 = 1 \\ dS_t &=& S_t' \left[b(t)dt + \sigma(t)dW_t\right], & S_0 = s_0 \end{array} \right. \tag{4.1}$$

where $s_0$ is a given vector in $\mathbb{R}^d_{++}$. We define the **market price of risk** by

$$\theta(t) = \sigma^{-1}(t)\left[b(t) - r(t)\mathbf{1}_d\right],$$

with $\mathbf{1}_d$ denoting the $d$-dimensional vector $\mathbf{1}_d \triangleq (1,1,\dots,1)'$.

We note that the equations in (4.1) specify a complete market model which, however, becomes incomplete by introducing a cone $\mathcal{K}$ of portfolio constraints, and in this case we have (see [KLSX91], p. 712; [CK92], p. 777; [KQ95], p. 50) that the set $\mathcal{Y}^{\mathcal{M}}$ of (2.10) satisfies

$$\mathcal{Y}^{\mathcal{M}} \subseteq \left\{Z_\nu(\cdot) \,:\, \nu\in\mathbb{K}, \text{ such that } Z_\nu(\cdot) \text{ is positive martingale}\right\}.$$

Here $\mathbb{K}$ is the set of all progressively measurable processes $\nu : [0,T]\times\Omega \to \mathbb{R}^d$, such that

$$\int_0^T ||\nu(t)||^2\, dt < \infty \quad \text{ and } \quad \nu(t)'p \geq 0,\ \forall\, p\in\mathcal{K},\, t\in[0,T]$$

hold almost surely (i.e., $\nu$ takes values in the barrier cone of $-\mathcal{K}$), and

$$Z_\nu(\cdot) \triangleq \exp\left(-\int_0^\cdot (\theta(t) + \sigma^{-1}(t)\nu(t))'\, dW_t - \frac{1}{2}\int_0^\cdot ||\theta(t)+\sigma^{-1}(t)\nu(t)||^2\, dt\right).$$

Let us recall also the $\sigma((\mathbb{L}^\infty)^*, \mathbb{L}^\infty)$-closure $\mathcal{D}$ of the set $\mathcal{M}$ in $(\mathbb{L}^\infty)^*$, as well as the enlargement $\mathcal{Y}^{\mathcal{D}}$ of $\mathcal{Y}^{\mathcal{M}}$ as in (2.10). In the following proposition we characterize the subset $\mathcal{Y}^{\mathcal{D}}_{\max}$ of the dual domain $\mathcal{Y}^{\mathcal{D}}$, consisting of processes that are strictly positive on the support $\operatorname{supp}\mu$ and *maximal* - i.e., not dominated by any other process in $\mathcal{Y}^{\mathcal{D}}$. We remind the reader that $\operatorname{supp}\mu$ is defined to be $[0,T]$ if $\mu$ charges $\{T\}$, and $[0,T)$ otherwise.

**Proposition 4.1.** *The elements of $\mathcal{Y}^{\mathcal{D}}_{\max}$ are local martingales of the form $\mathbb{P}[Y_t = Z_\nu(t),\ \forall\, t\in \operatorname{supp}\mu] = 1$ for some $\nu\in\mathbb{K}$.*

*Proof.* For simplicity, and without loss of generality, we shall assume in this proof that the market coëfficient processes $r$ and $b$ are identically equal to zero, that the volatility matrix $\sigma$ is the identity matrix, and that $\operatorname{supp}\mu = [0,T]$.

Let $Y^{\max} \in \mathcal{Y}^{\mathcal{D}}_{\max}$. The multiplicative decomposition theorem for positive special semimartingales (see [Jac79], Propositions 6.19 and 6.20) implies that there is continuous local martingale $M$ with $M_0 = 1$, and a nonincreasing predictable RCLL process $D$ with $D_0 = 1$ and $D_T > 0$ a.s., such that $Y^{\max}_t = M_t D_t$. By the martingale representation theorem for the Brownian filtration (see [KS91], Theorem 3.4.15 and Problem 3.4.16), there is a $d$-dimensional $\mathbb{F}$-progressively measurable process $\nu$ with $\int_0^T ||\nu(s)||^2 \, ds < \infty$ a.s. such that

$$M_t = \exp\left( - \int_0^t \nu(s)' \, dW(s) - \frac{1}{2} \int_0^t ||\nu(s)||^2 \, ds \right), \quad 0 \leq t \leq T.$$

For any admissible trading strategy $H$ and $x > 0$ such that

$$X^{x,H}_t \triangleq x + \int_0^t H(s)' \, dW(s) \geq 0, \quad \forall\, t \in [0,T]$$

holds almost surely, the process $Y X^{x,H}$ is a supermartingale by Theorem 2.10.

By Itô's formula (e.g. [Pro90], Section II.7) we have $d(Y_t X^{x,H}_t) = X^{x,H}_t \, dY_t + Y_{t-} \, dX^{x,H}_t + d[X^{x,H}, Y]_t$ and $dY_t = M_t \, dD_t + D_{t-} \, dM_t + d[M,D]_t$. Since $M$ is continuous and $D$ is predictable and of finite variation, $[M,D]_t \equiv M_0 D_0$, so $dY_t = M_t \, dD_t - D_{t-} M_t \nu(t) \, dW_t$, because $dM_t = -M_t \nu(t)' \, dW_t$. Furthermore, $dX^{x,H}_t = H'_t \, dW_t$, so $d[X^{x,H}, Y]_t = -D_{t-} M_t H'_t \nu(t) \, dt$. It follows that

$$d(Y_t X^{x,H}_t) = L_t + M_t \left[ X^{x,H}_t \, dD_t - D_{t-} H'_t \nu(t) \, dt \right], \tag{4.2}$$

where $L$ is a local martingale.

Now we prove that $\nu \in \mathbb{K}$. To do that, let us assume *per contra* that $\nu$ fails to satisfy the relation:

$$\nu(t)' p \geq 0 \;\text{ for all } p \in \mathcal{K}, \;\; \lambda \otimes \mathbb{P}\text{-a.s.} \tag{4.3}$$

Then, we can find a constant $\varepsilon > 0$, a predictable set $A$ such that $(\lambda \otimes \mathbb{P})(A) > 0$, and a bounded predictable process $\hat{H}$ taking values in $\mathcal{K}$, such that $\hat{H} = 0$ off $A$ and

$$D_{t-} \nu(t)' \hat{H}_t \leq -\varepsilon \quad \text{on } A. \tag{4.4}$$

We can also assume that $||\hat{H}_t|| = 1$ on $A$, $(\lambda \otimes \mathbb{P})$-a.s. For any $x > 0$, we define $S^x$ to be the first hitting time of the origin for the continuous process $X^{x,\hat{H}}$. Also, for $x > 0$ we define $H^x_t \triangleq \hat{H}_t \mathbf{1}_{[\![0,S^x]\!]}(t)$, so that $X^{x,H^x}_t \geq 0$ for all $t \in [0,T]$, a.s. Now we have all the ingredients to define a family of signed measures $\{\varphi_x\}_{x>0}$, given by

$$\varphi_x(B) \triangleq \mathbb{E} \left( \int_0^T \mathbf{1}_B(t) X^{x,H^x}_t \, dD_t + \varepsilon \int_0^T \mathbf{1}_B(t) \, dt \right), \tag{4.5}$$

on the $\mathbb{F}$-predictable subsets of $[0,T] \otimes \Omega$. By the supermartingale property of $Y X^{x,H^x}$, relations (4.2) and (4.4), and the strict positivity of the process $M$, we have that $\varphi_x(B) \leq 0$, for any $x > 0$ and any $\mathbb{F}$-predictable set $B \subseteq A \cap [\![0, S^x]\!]$. Due to the fact that $H^x$ is zero off $A$, $A \cap [\![0, S^x]\!]$ is still of positive $(\mu \otimes \mathbb{P})$-measure. By Theorem 2.1 of [DS95]), there exists an $\mathbb{F}$-predictable process $g : [0,T] \times \Omega \to \mathbb{R}$ and an $\mathbb{F}$-predictable set $N \subseteq [0,T] \times \Omega$ such that

$$D_t = \int_0^t g(u) \, du + \int_0^t \mathbf{1}_N(s) \, dD_u \quad \text{and} \quad \int_0^t \mathbf{1}_N(u) \, du = 0 \;\text{ for all } t \in [0,T] \tag{4.6}$$

hold almost surely, and $\int_0^T g(u)\,du \le D_T \le 1$, a.s. From the definition (4.5) of $\varphi_x$ and the decomposition (4.6), for any $x>0$ and any predictable $B \subseteq A \cap [\![0, S^x]\!] \setminus N$, we have

$$0 \quad \ge \quad \varphi_x(B) = \mathbb{E}\left(\int_0^T \left(X_t^{x,H^x} g(t) + \varepsilon\right) \mathbf{1}_B(t)\,dt\right).$$

The equation (4.6) states that $(\lambda \otimes \mathbb{P})(N) = 0$ for all $x>0$, so (4.7) implies that $X_t^{x,H^x} g(t) + \varepsilon \le 0$ holds $(\lambda \otimes \mathbb{P})$-a.e. on $A \cap [\![0, S^x]\!]$, for any $x>0$. We observe that the right-continuous inverse $Q^{-1}$ of the process $Q$ given by

$$Q_t \triangleq \int_0^t \mathbf{1}_A(s)\,ds = \int_0^t ||\hat{H}_s||^2\,ds = [X^{0,\hat{H}}, X^{0,\hat{H}}]_t, \quad 0 \le t \le T \tag{4.7}$$

is a random time-change which turns the process $X^{0,\hat{H}}$ into a Brownian motion $\xi_s \triangleq X^{0,\hat{H}}_{Q_s^{-1}}$ on the stochastic interval $\mathbb{S} \triangleq [\![0, Q_T)\!)$ (see Theorem 4.6, p. 174 and Problem 4.7, p. 175 in [KS91]). Let $f(s)$ be the composite process $g(Q_s^{-1})$, and let $R_x = Q_{S^x}$ be the hitting time of $-x$ by the Brownian motion $\xi$. Thus, for any $x>0$ and any $\mathbb{F}$-predictable set $B \subseteq \mathbb{S} \cap [\![0, R^x]\!]$, we have

$$1 \ge -\int_0^T \mathbf{1}_{A\cap[\![0,S^x]\!]}(u) g(u)\,du \ge -\int_0^{Q_T} \mathbf{1}_B(v) f(v)\,dv \ge \varepsilon \int_0^{Q_T} \mathbf{1}_B(v) \frac{1}{x+\xi_v}\,dv, \text{ a.s.} \tag{4.8}$$

The relation (4.8) implies that $x + B_s \ge \varepsilon$, $(\lambda \otimes \mathbb{P})$-a.e. on $\mathbb{S} \cap [\![0, R^x]\!]$. This is in contradiction with the fact that $\mathbb{P}(x + \xi_{R_x} = 0) > 0$ and, for small enough $x$, $\mathbb{P}(R_x \in \mathbb{S}) > 0$.

Therefore, the relation (4.3) holds, and we know that the process $M$ dominates $Y^{max}$. By truncation, $M$ can be obtained as the Fatou-limit of a sequence of martingales in $\mathcal{Y}^{\mathcal{M}}$, so by Theorem 2.9, $M \in \mathcal{Y}$. Theorem 2.10 states that $M$ is dominated by an element of $\mathcal{Y}^{\mathcal{D}}$. Since $M$ is a local martingale with $M_0 = 1$, and all elements $Y \in \mathcal{Y}^{\mathcal{D}}$ are supermartingales with $Y_0 \le 1$, we can find a sequence $\{T_n\}_{n\in\mathbb{N}}$ of stopping times that reduces $M$, and use it to conclude that $M \in \mathcal{Y}^{\mathcal{D}}_{\max} \subseteq \mathcal{Y}^{\mathcal{D}}$ and $Y^{\max} = M$. □

Because of the fact that the optimal solution of the dual problem must be positive on the supp $\mu$, we have the following:

**Corollary 4.2.** *In the setting of an Itô-process market, the primal problem admits a unique solution,*

$$c(t) = I(t, yZ_\nu(t)D_t), \quad 0 \le t \le T, \tag{4.9}$$

*for some constant $y>0$, and some predictable process $\nu$ that takes values in the barrier cone of $-\mathcal{K}$ and satisfies $\int_0^T ||\nu(s)||^2\,ds < \infty$ a.s., and some positive, nonincreasing and $\mathcal{F}$-predictable process $D$ with $D_0 \le 1$. Both processes $Z_\nu$ and $Z_\nu D$ are in $\mathcal{Y}^{\mathcal{D}}$.*

*Remark* 9. When the market is complete, or, more generally, when the terminal value of the endowment process is "attainable" (i.e., $x + \mathcal{E}_T = X_T$ for some $X \in \mathcal{X}$ as in (2.2), then the dual objective function $\mathbb{Q} \mapsto J(y, \mathbb{Q})$ of (3.3) is monotone in $Y^{\mathbb{Q}}$ and thus the optimal solution takes the form $c(t) = I(t, yZ_\nu(t))$, $0 \le t \le T$, with $D \equiv 1$ in (4.9).

4.2. **Optimal Consumption of a Random Endowment.** In this example we consider a situation in which the agent must optimally distribute an unknown future endowment without any possibility of hedging the uncertainty in a financial market. This problem was studied by Lakner and Slud in [LS91] in a point-process setting. We shall consider the following version of it:

**Problem 4.3.** Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses, and let $\varepsilon(\cdot)$ be a nonnegative progressively measurable process such that $\mathcal{E}_T = \int_0^T \varepsilon(t)\,dt$ is uniformly bounded from above and away from the origin. With $U$, a given utility function, the question is to find a progressively measurable, nonnegative consumption-rate process $c(\cdot)$ satisfying $\int_0^T c(t)\,dt < \infty$ a.s. - so as to maximize the expected utility $\mathbb{E}\int_0^T U(c(t))\,dt$, subject to the constraint

$$\int_0^T c(t)\,dt \leq \int_0^T \varepsilon(t)\,dt \quad \text{a.s.} \tag{4.10}$$

The following theorem was proved in [LS91]. As usual, $I(\cdot)$ will denote the inverse marginal utility, i.e. $I(y) = (U')^{-1}(y)$, for $0 < y < \infty$. We include a proof for the reader's convenience.

**Theorem 4.4.** *Suppose there exists a positive $\mathbb{F}$-martingale $Y$ such that*

$$\int_0^T I(Y_t)\,dt \;=\; \int_0^T \varepsilon(t)\,dt\,, \qquad a.s. \tag{4.11}$$

*Then an optimal consumption process is given by*

$$\hat{c}(t) = I(Y_t), \quad 0 \leq t \leq T. \tag{4.12}$$

*Proof.* ¿From the inequality $U(I(y)) \geq U(c) + yI(y) - yc$, valid for $y > 0$, and $c > 0$, we obtain

$$U(I(Y_t)) \geq U(c(t)) + Y_t I(Y_t) - Y_t c(t),$$

for every positive, adapted process $\{c(t),\, t \in [0,T]\}$. Therefore,

$$\mathbb{E}\int_0^T U(\hat{c}(t))\,dt \;\geq \mathbb{E}\int_0^T U(c(t))\,dt \,+\, \mathbb{E}\int_0^T Y_t I(Y_t)\,dt \,-\, \mathbb{E}\int_0^T Y_t c(t)\,dt,$$

and the optimality of the process $\hat{c}$ in (4.12), amongst those that satisfy (4.10), will follow once we have shown that this latter constraint implies

$$\mathbb{E}\int_0^T Y_t I(Y_t)\,dt \;\geq\; \mathbb{E}\int_0^T Y_t c(t)\,dt. \tag{4.13}$$

To do that, it suffices to introduce the probability measure $\tilde{\mathbb{P}}(A) \triangleq \frac{1}{y}\mathbb{E}[Y_T \mathbf{1}_A]$ for $A \in \mathcal{F}_T$, where $y = \mathbb{E}[Y_0] \in (0,\infty)$. This measure is equivalent to $\mathbb{P}$, and thus the martingale property of $Y$, (4.10) and (4.11) lead to

$$\mathbb{E}\int_0^T Y_t I(Y_t)\,dt \;=\; y\,\tilde{\mathbb{E}}\int_0^T I(Y_t)\,dt \;=\; y\,\tilde{\mathbb{E}}\int_0^T \varepsilon(t)\,dt \;=\; \mathbb{E}\int_0^T Y_t\,\varepsilon(t)\,dt \geq= \;\mathbb{E}\int_0^T Y_t\,c(t)\,dt,$$

which is (4.13). □

We prove the following existence result, which is a partial converse of Theorem 4.4:

**Proposition 4.5.** *When the utility function $U(\cdot)$ satisfies the "reasonable asymptotic elasticity" condition of Definition 3.1 (4), the optimization Problem 4.3 has a unique solution which is of the form $\hat{c}(t) = I(Y_t),\ 0 \le t \le T$, for some positive, RCLL supermartingale $Y$; this process satisfies*

$$\int_0^T I(Y_t)\,dt = \int_0^T \varepsilon(t)\,dt\,, \quad a.s. \tag{4.14}$$

*Proof.* We note first that Problem 4.3 is a special case of our Primal problem with a one-dimensional *"stock price"* process $S_t \equiv 1$ and trivial bond-price process $B_t \equiv 1$. In this case *all* measures equivalent to $\mathbb{P}$ are equivalent supermartingale measures, and by Theorem 2.10 any RCLL-supermartingale $Y$ with $Y_0 \le 1$ is in $\mathcal{Y}$. By the Main Theorem 3.10, the unique optimal consumption-rate process is given by $\hat{c}(t) = I(yY_t^{\hat{\mathbb{Q}}}),\ \ 0 \le t \le T$, for some $y > 0$ and some $\hat{\mathbb{Q}} \in \mathcal{D}$. To finish the proof we define $Y_t = yY_t^{\hat{\mathbb{Q}}}$, and note that Proposition 2.12 implies that

$$\int_0^T \hat{c}(t)\,dt \quad \le \quad \int_0^T \varepsilon(t)\,dt\,, \quad \text{a.s.} \tag{4.15}$$

because every measure equivalent to $\mathbb{P}$ is in $\mathcal{M}$. From the Main Theorem 3.10 (v) and (vi), it follows that, for the optimal solution $\hat{\mathbb{Q}} \in \mathcal{D}$ of the dual problem, we have

$$\mathbb{E}\int_0^T Y_T^{\hat{\mathbb{Q}}}\hat{c}(t)\,dt \;=\; \left\langle \hat{\mathbb{Q}}, \int_0^T \hat{c}(t)\,dt \right\rangle \;\ge\; \left\langle \hat{\mathbb{Q}}, \int_0^T \varepsilon(t)\,dt \right\rangle \;=\; \mathbb{E}\int_0^T Y_T^{\hat{\mathbb{Q}}}\varepsilon(t)\,dt\,. \tag{4.16}$$

The random variable $Y_T^{\hat{\mathbb{Q}}} = L_T^{\hat{\mathbb{Q}}} = d(\hat{\mathbb{Q}})^r/d\mathbb{P}$ is strictly positive, so the equation (4.14) follows from (4.15) and (4.16). $\square$

## Appendix A. Proof of the main theorem 3.10

In this part we state and prove a number of results leading to the proof of our Main Theorem 3.10. To simplify the notation we do not relabel the indices when passing to a subsequence.

A.1. **Existence in the Dual Problem.** We study the dual problem first. In this subsection we point out some properties of the dual objective function and establish the existence of $\hat{\mathbb{Q}} \in \mathcal{D}$ which is optimal in the dual problem of (3.3). The negative part $\max\{0, -V\}$ of the random field $V$ will be denoted by $V^-(\cdot)$. Our first result establishes a lower-semicontinuity property for the nonlinear part of the dual objective function. We remind the reader that $V$ is the convex cunjugate of $U$ introduced in (3.1).

**Lemma A.1.** *For $y > 0$, the family of random processes $\{V^-(\cdot, yY_\cdot^{\mathbb{Q}}) : \mathbb{Q} \in \mathcal{D}\}$ is uniformly integrable with respect to the product measure $(\mu \otimes \mathbb{P})$ on $[0,T] \times \Omega$. Furthermore, the lower- semicontinuity relation*

$$\mathbb{E}\left[\int_0^T V(t, yY_t^{\mathbb{Q}})\,\mu(dt)\right] \le \liminf_n \mathbb{E}\left[\int_0^T V(t, yY_t^{\mathbb{Q}^{(n)}})\,\mu(dt)\right] \tag{A.1}$$

*holds for all sequences $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$ such that $Y^{\mathbb{Q}^{(n)}}$ converges to a RCLL supermartingale $Y^{\mathbb{Q}}$, $(\mu \otimes \mathbb{P})$-a.e.*

*Proof.* Let $\underline{V}(\cdot)$ be a minorant of $V(\cdot,\cdot)$, as introduced in Definition 3.6. We define $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ to be the right-continuous inverse of $\underline{V}^-(\cdot)$, i.e. $\varphi(x)\triangleq\inf\left\{y\geq 0\,:\,\underline{V}^-(y)<x\right\}$, for $x\geq 0$. Suppose first that $\varphi(x)$ is finite for all $x\geq 0$. Then, by L'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\varphi(x)}{x}=\lim_{x\to\infty}\varphi'(x)=\lim_{y\to\infty}\frac{1}{(\underline{V}^-)'(y)}=\infty.$$

The family $\left\{\varphi(V^-(\cdot,yY^{\mathbb{Q}}_\cdot))\,:\,\mathbb{Q}\in\mathcal{D}\right\}$ is bounded in $L^1(\mu\otimes\mathbb{P})$, because

$$\mathbb{E}\left[\int_0^T\varphi(V^-(t,yY^{\mathbb{Q}}_t))\,\mu(dt)\right]\;\leq\;\mathbb{E}\left[\int_0^T\varphi(\underline{V}^-(yY^{\mathbb{Q}}_t))\,\mu(dt)\right]\;\leq\;\varphi(0)+\mathbb{E}\left[\int_0^T yY^{\mathbb{Q}}_t\,\mu(dt)\right]\;\leq\;\varphi(0)+y.$$

Thus, by the thorem of de la Vallé Poussin (see [Shi96], Lemma II.6.3. p. 190), the family of random variables $\left\{\varphi(V^-(\cdot,yY^{\mathbb{Q}}_t))\,:\,\mathbb{Q}\in\mathcal{D}\right\}$ is uniformly integrable. If $\varphi(x)=\infty$, for some $x>0$, then $\underline{V}^-(\cdot)$ is a bounded function and uniform integrability follows readily.

Let $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}}\subseteq\mathcal{D}$ be a sequence such that $\{Y^{\mathbb{Q}^{(n)}}\}_{n\in\mathbb{N}}$ converges to a RCLL-supermartingale $Y^{\mathbb{Q}}$, $(\mu\otimes\mathbb{P})$-a.e. By uniform integrability we have that

$$\mathbb{E}\left[\int_0^T V^-(t,yY^{\mathbb{Q}^{(n)}}_t)\,\mu(dt)\right]\longrightarrow\mathbb{E}\left[\int_0^T V^-(t,yY^{\mathbb{Q}}_t)\,\mu(dt)\right],\ \text{as } n\to\infty. \tag{A.2}$$

As for the positive parts, Fatou's lemma gives that

$$\liminf_n\mathbb{E}\left[\int_0^T V^+(t,yY^{\mathbb{Q}^{(n)}}_t)\,\mu(dt)\right]\;\geq\;\mathbb{E}\left[\int_0^T V^+(t,yY^{\mathbb{Q}}_t)\,\mu(dt)\right]. \tag{A.3}$$

The claim now follows from (A.2) and (A.3). □

The following result establishes the existence of a solution to the dual problem.

**Proposition A.2.** *For each $y>0$ such that $\mathfrak{V}(y)<\infty$, there is $\hat{\mathbb{Q}}\in\mathcal{D}$ such that*

$$\mathfrak{V}(y)\;=\;J(y,\hat{\mathbb{Q}})\;\;=\mathbb{E}\left[\int_0^T V(t,yY^{\hat{\mathbb{Q}}}_t)\,\mu(dt)\right]+y\langle\hat{\mathbb{Q}},\mathcal{E}_T\rangle.$$

*Proof.* We fix $y>0$ and let $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}}$ be a minimizing sequence for $J(y,\cdot)$. We first assume that the sequence $\{\langle\mathbb{Q}^{(n)},\mathcal{E}_T\rangle\}_{n\in\mathbb{N}}$ converges in $\mathbb{R}$. This can be justified by extracting a subsequence if necessary. By Lemma 5.2 in [FK97] we can find a sequence of convex combinations $\{Y^{\mathbb{Q}^{(n)}}_\cdot\}_{n\in\mathbb{N}}$ and a RCLL-supermartingale $Y$ such that $\{Y^{\mathbb{Q}^{(n)}}_\cdot\}_{n\in\mathbb{N}}$ converges towards $Y$ in the Fatou sense. Because of boundedness in $\mathbb{L}^1(\mu\otimes\mathbb{P})$, thanks to Komlós's theorem we can pass to a further sequence of convex combinations to achieve convergence $(\mu\otimes\mathbb{P})$-a.e. By Proposition 2.6, the limit is still $Y$. Because of the convexity of $V(t,\cdot)$ and the convergence of the sequence $\{\langle\mathbb{Q}^{(n)},\mathcal{E}_T\rangle\}_{n\in\mathbb{N}}$, passing to convex combinations preserves the property of being a minimizing sequence. By Proposition 2.6, the limit $Y$ is of the form $Y^{\hat{\mathbb{Q}}}$ for some (and then every) cluster point $\hat{\mathbb{Q}}$ of $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}}$; the existence of such a cluster point is guaranteed by Alaoglu's theorem. Invoking Lemma A.1 establishes the claim of the proposition. □

A.2. **Conjugacy and finiteness of $\mathfrak{U}(\cdot)$ and $\mathfrak{V}(\cdot)$.** The next step is to establish a conjugacy relation between $\mathfrak{U}(\cdot)$ and $\mathfrak{V}(\cdot)$. The most important tool in this endeavor is the Minimax Theorem.

**Lemma A.3.** *The function $\mathfrak{V}(\cdot)$ is the convex conjugate of $\mathfrak{U}(\cdot)$, i.e.*

$$\mathfrak{V}(y) = \sup_{x>0}[\mathfrak{U}(x) - xy] \quad \text{for } y > 0.$$

*Proof.* For fixed $y \in (0,\infty)$ and $n \in \mathbb{N}$, let $\mathcal{S}_n$ denote the set of all nonnegative, progressively measurable processes $c : [0,T] \times \Omega \to [0,n]$. The sets $\mathcal{S}_n$ can be viewed as a closed subsets of balls in $\mathbb{L}^\infty(\mu \otimes \mathbb{P})$. Thanks to the concavity of $U(t,\cdot)$, the compactness of $\mathcal{S}_n$ (by Alaoglu's theorem; see [Woj96], Theorem 2.A.9), and the convexity of $\mathcal{D}$, we can use the Minimax Theorem (see [Str85], Theorem 45.8 and its corollaries) to obtain

$$\sup_{c\in\mathcal{S}_n} \inf_{\mathbb{Q}\in\mathcal{D}} \left( \mathbb{E} \int_0^T \left( U(t,c(t)) - yY_t^{\mathbb{Q}} c(t) \right) \mu(dt) + y\langle \mathbb{Q}, \mathcal{E}_T\rangle \right) =$$

$$\inf_{\mathbb{Q}\in\mathcal{D}} \sup_{c\in\mathcal{S}_n} \left( \mathbb{E} \int_0^T \left( U(t,c(t)) - yY_t^{\mathbb{Q}} c(t) \right) \mu(dt) + y\langle \mathbb{Q}, \mathcal{E}_T\rangle \right), \tag{A.4}$$

for any $n \in \mathbb{N}$, $y > 0$. Proposition 2.12 guarantees that $\cup_{x>0}\mathcal{A}^\mu(x+\mathcal{E}) = \cup_{x>0}(\mathcal{A}^\mu)'(x+\mathcal{E})$ where

$$(\mathcal{A}^\mu)'(x+\mathcal{E}) \triangleq \left\{ c \in \mathcal{A}^\mu(x+\mathcal{E}) : \sup_{\mathbb{Q}\in\mathcal{D}} \left( \mathbb{E}\int_0^T c(t) Y_t^{\mathbb{Q}}\, \mu(dt) - \langle \mathbb{Q}, \mathcal{E}_T\rangle \right) = x \right\}.$$

Thus, by pointwise approximation of elements of $\cup_{x>0}(\mathcal{A}^\mu)'(x+\mathcal{E})$ by elements of $\cup_{n\in\mathbb{N}}\mathcal{S}_n$, we obtain

$$\lim_{n\to\infty} \sup_{c\in\mathcal{S}_n} \inf_{\mathbb{Q}\in\mathcal{D}} \left( \mathbb{E} \int_0^T \left( U(t,c(t)) - yY_t^{\mathbb{Q}} c(t) \right) \mu(dt) + y\langle \mathbb{Q}, \mathcal{E}_T\rangle \right) =$$

$$= \sup_{x>0} \sup_{c\in(\mathcal{A}^\mu)'(x+\mathcal{E})} \mathbb{E}\left[ \int_0^T \left( U(t,c(t)) - xy \right) \mu(dt) \right] = \sup_{x>0}[\mathfrak{U}(x) - xy]. \tag{A.5}$$

We define $V^{(n)}(t,y) \triangleq \sup_{0<x\le n}[U(t,x) - xy]$, and the pointwise maximization yields

$$\inf_{\mathbb{Q}\in\mathcal{D}} \sup_{c\in\mathcal{S}_n} \left( \mathbb{E} \int_0^T \left( U(t,c(t)) - yY_t^{\mathbb{Q}} c(t) \right) \mu(dt) + y\langle \mathbb{Q}, \mathcal{E}_T\rangle \right)$$

$$= \inf_{\mathbb{Q}\in\mathcal{D}} \left( \mathbb{E} \int_0^T V^{(n)}(t, yY_t^{\mathbb{Q}})\, \mu(dt) + y\langle \mathbb{Q}, \mathcal{E}_T\rangle \right) \triangleq \mathfrak{V}^{(n)}(y) \tag{A.6}$$

From (A.4), (A.5) and (A.6) we conclude that $\lim_n \mathfrak{V}^{(n)}(y) = \sup_{x>0}[\mathfrak{U}(x) - xy]$. To prove the claim of the lemma it is enough to show that $\lim_{n\to\infty} \mathfrak{V}^{(n)}(y) \geq \mathfrak{V}(y)$, since $\mathfrak{V}^{(n)}(y) \leq \mathfrak{V}(y)$ holds for all $y > 0$, $n \in \mathbb{N}$. For a fixed $y > 0$, let $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{D}$ be a sequence such that

$$\lim_{n\to\infty} \left( \mathbb{E} \int_0^T V^{(n)}(t, yY_t^{\mathbb{Q}^{(n)}})\, \mu(dt) + y\langle \mathbb{Q}^{(n)}, \mathcal{E}_T\rangle \right) = \lim_{n\to\infty} \mathfrak{V}^{(n)}(y).$$

Using the construction from Lemma A.1 we can assume that $\langle \mathbb{Q}^{(n)}, \mathcal{E}_T\rangle \to \langle \mathbb{Q}^*, \mathcal{E}_T\rangle$ and that $Y^{\mathbb{Q}^{(n)}} \to Y^{\mathbb{Q}^*}$ as $n \to \infty$, both in the $(\mu \otimes \mathbb{P})$-a.e. and in the Fatou sense, where $\mathbb{Q}^*$ is a cluster point of $\{\mathbb{Q}^{(n)}\}_{n\in\mathbb{N}}$.

Let $\overline{U}(\cdot)$ be a majorant of $U$, and $\overline{V}(\cdot)$ its conjugate. Then it is easy to see that

$$V^{(n)}(t,y) \leq \overline{V}^{(n)}(y) := \sup_{0<x\leq n} [\overline{U}(x) - xy], \text{ for all } t \in [0,T]$$

and $\overline{V}^{(n)}(y) = \overline{V}(y)$ for $y \geq \overline{I}(1) \geq \overline{I}(n)$ where $\overline{I}(y) := (\overline{U}'(\cdot))^{-1}(y)$. The argument from Lemma A.1 takes care of the uniform integrability of the sequence of processes $\{V^{(n)}(\cdot, Y_\cdot^{\mathbb{Q}^{(n)}})^-\}_{n\in\mathbb{N}}$ as well as of the following chain of inequalities

$$\lim_{n\to\infty}\left(\mathbb{E}\int_0^T V^{(n)}(t, Y_t^{\mathbb{Q}^{(n)}})\,\mu(dt) + y\langle \mathbb{Q}^{(n)}, \mathcal{E}_T\rangle\right) \geq \left(\mathbb{E}\int_0^T V(t, Y_t^{\mathbb{Q}^*})\,\mu(dt) + y\langle \mathbb{Q}^*, \mathcal{E}_T\rangle\right) \geq \mathfrak{V}(y),$$

settling the claim of the lemma. □

*Remark* 10. It is a consequence of the decrease of $\mathfrak{V}(\cdot)$ and the preservation of properness in the conjugacy relation (see [Roc70], Theorem 12.2, p. 104 ) that the Standing Assumption 3.9 implies the existence of $y_0 > 0$ such that $\mathfrak{V}(y) < \infty$ for $y > y_0$. Furthermore, the strict convexity of $V(t,\cdot)$ allows us to denote by $\hat{\mathbb{Q}}^y$ the unique (as far as its action on $\mathcal{E}_T$ and the corresponding supermartingale $Y^{\hat{\mathbb{Q}}^y}$ are concerned) minimizer of the dual problem for $y$ such that $\mathfrak{V}(y) < \infty$.

**Lemma A.4.** $\mathfrak{V}(y) \in (-\infty, \infty)$ *for all* $y > 0$.

*Proof.* Let $\underline{U}(\cdot)$ be a minorant of $U(\cdot,\cdot)$. $\underline{U}(\cdot)$ is a utility function and the convex conjugate $\underline{V}(\cdot)$ of $\underline{U}(\cdot)$ satisfies $\underline{V}(y) \leq V(t,y)$ for all $t$. Let $\rho = ||\mathcal{E}_T||_{\mathbb{L}^\infty}$. By the convexity of $\underline{V}(\cdot)$ and Jensen's inequality, we have

$$\begin{aligned}\mathfrak{V}(y) &= \inf_{\mathbb{Q}\in\mathcal{D}}\left(\mathbb{E}\left[\int_0^T V(t, yY_t^{\mathbb{Q}})\,\mu(dt)\right] + y\langle \mathbb{Q}, \mathcal{E}_T\rangle\right) \geq \inf_{\mathbb{Q}\in\mathcal{D}} \mathbb{E}\left[\int_0^T \underline{V}(yY_t^{\mathbb{Q}})\,\mu(dt)\right] \\ &\geq \inf_{\mathbb{Q}\in\mathcal{D}} \underline{V}\left(\mathbb{E}\left[\int_0^T yY_t^{\mathbb{Q}}\,\mu(dt)\right]\right) \geq \underline{V}(y) > -\infty. \end{aligned} \tag{A.7}$$

To prove that $\mathfrak{V}(y)$ is finite, we first choose $y > 0$ such that $\mathfrak{V}(y) < \infty$ – its existence is guaranteed by Remark 10. For some $\gamma \in \Gamma_3 \cap [\mathrm{A}E[U], 1)$ a.s, and some $0 < \rho < 1$, Proposition 3.7 implies that there exists $y_0 > 0$ such that

$$V(t, \rho y) \leq C\, V(t,y), \text{ for } y \leq y_0$$

where $C = \rho^{-\gamma/(1-\gamma)}$. By Proposition A.2 there is $\hat{\mathbb{Q}}^y \in \mathcal{D}$ such that $\mathfrak{V}(y) = \mathbb{E}\left[\int_0^T V(t, yY_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right]$, so

$$\begin{aligned}\mathfrak{V}(\rho y) &\leq \mathbb{E}\left[\int_0^T V(t, \rho y Y_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right] \\ &= \mathbb{E}\left[\int_0^T V(t, \rho y Y_t^{\hat{\mathbb{Q}}^y})\mathbf{1}_{\left\{\rho y Y_t^{\hat{\mathbb{Q}}^y} > y_0\right\}}\,\mu(dt)\right] + \mathbb{E}\left[\int_0^T V(t, \rho y Y_t^{\hat{\mathbb{Q}}^y})\mathbf{1}_{\left\{\rho y Y_t^{\hat{\mathbb{Q}}^y} \leq y_0\right\}}\,\mu(dt)\right] \\ &\leq \sup_t V(t, y_0) + C\mathbb{E}\left[\int_0^T V(t, yY_t^{\hat{\mathbb{Q}}^y})\mathbf{1}_{\left\{\rho y Y_t^{\hat{\mathbb{Q}}^y} \leq y_0\right\}}\,\mu(dt)\right] < \infty.\end{aligned}$$

We conclude that $\mathfrak{V}(y) < \infty$ for all $y > 0$, due to the decrease of $\mathfrak{V}(\cdot)$. □

Having established the existence and essential uniqueness of the solution, and the finiteness of the value function for the dual problem, we can apply ideas from the calculus of variations to obtain the following:

**Lemma A.5.** *For each $y > 0$ and each $\mathbb{Q} \in \mathcal{D}$ we have*

$$\mathbb{E}\left[\int_0^T (Y_t^{\mathbb{Q}} - Y_t^{\hat{\mathbb{Q}}^y}) I(t, yY_t^{\hat{\mathbb{Q}}^y})\, \mu(dt)\right] + \langle \hat{\mathbb{Q}}^y - \mathbb{Q}, \mathcal{E}_T\rangle \;\leq\; 0,$$

*where $\hat{\mathbb{Q}}^y$ is the optimal solution to the dual problem of (3.3) (as in Proposition A.2 and Remark 10).*

*Proof.* For $y > 0$, $\varepsilon \in (0,1)$ and $\mathbb{Q}^\varepsilon = (1-\varepsilon)\hat{\mathbb{Q}}^y + \varepsilon\mathbb{Q}$, the optimality of $\hat{\mathbb{Q}}^y$ implies

$$\begin{aligned}
0 \;&\leq\; \mathbb{E}\left[\int_0^T \left(V(t, yY_t^{\mathbb{Q}^\varepsilon}) - V(t, yY_t^{\hat{\mathbb{Q}}^y})\right) \mu(dt)\right] + y\langle \mathbb{Q}^\varepsilon - \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle \\
&\leq\; \mathbb{E}\left[\int_0^T y(Y_t^{\hat{\mathbb{Q}}^y} - Y_t^{\mathbb{Q}^\varepsilon}) I(t, yY_t^{\mathbb{Q}^\varepsilon})\, \mu(dt)\right] + y\langle \mathbb{Q}^\varepsilon - \hat{\mathbb{Q}}, \mathcal{E}_T\rangle \\
&=\; \varepsilon y \left(\mathbb{E}\left[\int_0^T (Y_t^{\hat{\mathbb{Q}}^y} - Y_t^{\mathbb{Q}}) I(t, yY_t^{\mathbb{Q}^\varepsilon})\, \mu(dt)\right] + \langle \hat{\mathbb{Q}}^y - \mathbb{Q}, \mathcal{E}_t\rangle\right).
\end{aligned}$$

Since

$$\left((Y_t^{\mathbb{Q}} - Y_t^{\hat{\mathbb{Q}}^y}) I(t, yY_t^{\mathbb{Q}^\varepsilon})\right)^- \;\leq\; Y_t^{\hat{\mathbb{Q}}^y} I(t, yY_t^{\mathbb{Q}^\varepsilon}) \;\leq\; Y_t^{\hat{\mathbb{Q}}^y} I(t, y(1-\varepsilon)Y_t^{\hat{\mathbb{Q}}^y}),$$

we can follow the same reasoning as in Lemma A.4 to show that the last term is dominated by an random process on $\Omega \times [0,T]$ which is $(\mu \otimes \mathbb{P})$-integrable. Now we can let $\varepsilon \to 0$ and apply Fatou's lemma, to obtain the stated inequality. □

A.3. **Differentiability of the value functions.** We turn our attention the the differentiability properties of the value functions.

**Proposition A.6.** *The dual value function $\mathfrak{V}(\cdot)$ is strictly convex and continuously differentiable on $\mathbb{R}_+$; its derivative is given by*

$$\mathfrak{V}'(y) = \langle \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle - \mathbb{E}\left[\int_0^T Y^{\hat{\mathbb{Q}}^y} I(t, yY^{\hat{\mathbb{Q}}^y})\, \mu(dt)\right].$$

*Proof.* The fact that $\mathfrak{V}(\cdot)$ is strictly convex follows from the strict convexity of $V(t,\cdot)$. Therefore, to show that $\mathfrak{V}(\cdot)$ is continuously differentiable, it is enough (by convexity) to show that its derivative exists everywhere on $(0,\infty)$. We start by fixing $y > 0$, and defining the function

$$h(z) \triangleq \mathbb{E}\left[\int_0^T V(t, zY_t^{\hat{\mathbb{Q}}^y})\, \mu(dt)\right] + z\langle \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle, \quad z > 0$$

This function is convex and, by definition of the optimal solution $\hat{\mathbb{Q}}^y$ of the dual problem, we have $h(z) \geq \mathfrak{V}(z)$ for all $z > 0$ and $h(y) = \mathfrak{V}(y)$. Again by convexity, we obtain

$$\Delta^- h(y) \leq \Delta^- \mathfrak{V}(y) \leq \Delta^+ \mathfrak{V}(y) \leq \Delta^+ h(y),$$

where $\Delta^+$ and $\Delta^-$ denote right- and left-derivatives, respectively. Now

$$\begin{aligned}\Delta^+ h(y) &= \lim_{\varepsilon\to 0}\frac{h(y+\varepsilon)-h(y)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\,\mathbb{E}\left[\int_0^T V(t,(y+\varepsilon)Y_t^{\hat{\mathbb{Q}}^y}) - V(t,yY_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right] + \langle\hat{\mathbb{Q}}^y,\mathcal{E}_T\rangle\\ &\le \liminf_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon}\right)\mathbb{E}\left[\int_0^T \varepsilon Y_t^{\hat{\mathbb{Q}}^y} I(t,(y+\varepsilon)Y_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right] + \langle\hat{\mathbb{Q}}^y,\mathcal{E}_T\rangle\\ &= -\mathbb{E}\left[\int_0^T Y_t^{\hat{\mathbb{Q}}^y} I(t,yY_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right] + \langle\hat{\mathbb{Q}}^y,\mathcal{E}_T\rangle\end{aligned}$$

by the Monotone Convergence Theorem. Similarly, we get

$$\Delta^- h(y) \ge \limsup_{\varepsilon\to 0}\mathbb{E}\left[-\int_0^T Y_t^{\hat{\mathbb{Q}}^y} I(t,(y-\varepsilon)Y_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right] + <\hat{\mathbb{Q}}^y,\mathcal{E}_t> .$$

Let $y_0$ be the constant from $\Gamma_4$, Lemma 3.7, corresponding to some $\mathrm{AE}[U]\le\gamma<1$ a.s. Then

$$|Y_t^{\hat{\mathbb{Q}}^y} I(t,(y-\varepsilon)Y_t^{\hat{\mathbb{Q}}^y})| \le |Y_t^{\hat{\mathbb{Q}}^y} I(t,(y-\varepsilon)Y_t^{\hat{\mathbb{Q}}^y})|\,\mathbf{1}_{\left\{Y_t^{\hat{\mathbb{Q}}^y}\le y_0/y\right\}} + |Y_t^{\hat{\mathbb{Q}}^y} I(t,(y-\varepsilon)Y_t^{\hat{\mathbb{Q}}^y})|\,\mathbf{1}_{\left\{Y_t^{\hat{\mathbb{Q}}^y}> y_0/y\right\}}.$$

We fix $\varepsilon_0$ and observe that for $\varepsilon<\varepsilon_0$, by Lemma 3.7, the second part is dominated by

$$\frac{1}{y-\varepsilon_0}\frac{\gamma}{1-\gamma}V(t,(y-\varepsilon_0)Y_t^{\hat{\mathbb{Q}}^y}) \le \frac{1}{y-\varepsilon_0}\frac{\gamma}{1-\gamma}CV(t,yY_t^{\hat{\mathbb{Q}}^y}), \tag{A.8}$$

for some constant $C$. This last expression is in $L^1(\mu\otimes\mathbb{P})$, by finiteness of $\mathfrak{V}(\cdot)$. On the other hand, the first part in (A.8) is dominated by $K_1(\frac{y-\varepsilon_0}{y}y_m)Y_t^{\hat{\mathbb{Q}}^y}$, which is in $L^1(\mu\otimes\mathbb{P})$ by the supermartingale property of $Y^{\hat{\mathbb{Q}}^y}$. Having prepared the ground for the Dominated Convergence Theorem, we can let $\varepsilon\to 0$ and obtain

$$\Delta^- h(y) \ge <\hat{\mathbb{Q}}^y,\mathcal{E}_T> -\mathbb{E}\left[\int_0^T Y_t^{\hat{\mathbb{Q}}^y} I(t,yY_t^{\hat{\mathbb{Q}}^y})\,\mu(dt)\right],$$

completing the proof of the proposition. □

**Lemma A.7.** *The dual value function $\mathfrak{V}(\cdot)$ has the following asymptotic behavior:*

(i) $\mathfrak{V}'(0+)=-\infty$,

(ii) $\mathfrak{V}'(\infty)\in[\inf_{\mathbb{Q}\in\mathcal{D}}<\mathbb{Q},\mathcal{E}_T>,\sup_{\mathbb{Q}\in\mathcal{D}}<\mathbb{Q},\mathcal{E}_T>]$.

*Proof.*

(i) Suppose first there is a minorant $\underline{V}(\cdot)$ of $V(\cdot,\cdot)$ such that $\underline{V}(0+)=\infty$. Letting $y\to 0$ in (A.7), we get $\mathfrak{V}(0+)=\infty$ and, by convexity, $\mathfrak{V}'(0+)=-\infty$.

In the case when $\underline{V}(0+)<\infty$ for each minorant $\underline{V}(\cdot)$ of $V(\cdot,\cdot)$, we can easily construct a majorant $\overline{V}(\cdot)$ such that $\overline{V}(0+)<\infty$, using the properties of finctions $K_1$ and $K_2$ from Definition 3.1. We pick such a majorant $\overline{V}(\cdot)$, a minorant $\underline{V}(\cdot)$, set $\overline{I}(\cdot)=-\overline{V}'(\cdot)$, $D=\overline{V}(0+)-\underline{V}(0+)$, and choose $\mathbb{Q}\in\mathcal{D}$. Then, with $\rho=||\mathcal{E}_T||_{\mathbb{L}^\infty}$,

$$\begin{aligned}-\mathfrak{V}'(y) &\ge \frac{\mathfrak{V}(0+)-\mathfrak{V}(y)}{y} \ge \frac{1}{y}\left[(\underline{V}(0+)-\overline{V}(0+))+\overline{V}(0+)-\mathfrak{V}(y)\right]\\ &\ge \frac{-D-\rho y}{y}+\frac{\overline{V}(0+)-\mathbb{E}\left[\int_0^T\overline{V}(yY_t^{\mathbb{Q}})\,\mu(dt)\right]}{y}\\ &\ge \frac{-D-\rho y}{y}+\mathbb{E}\left[\int_0^T Y_t^{\mathbb{Q}}\overline{I}(yY_t^{\mathbb{Q}})\,\mu(dt)\right]\longrightarrow\infty,\quad\text{as } y\to\infty,\end{aligned}$$

by the Monotone convergence theorem.

(ii) By l'Hôpital's rule we have

$$\begin{aligned}\mathfrak{V}'(\infty) &= \lim_{y\to\infty}\frac{\mathfrak{V}(y)}{y} = \lim_{y\to\infty}\frac{\inf_{\mathbb{Q}\in\mathcal{D}}\left(\mathbb{E}\left[\int_0^T V(t,yY_t^{\mathbb{Q}})\,\mu(dt)\right]+y<\mathbb{Q},\mathcal{E}_T>\right)}{y}\\ &\in \left[L+\inf_{\mathbb{Q}\in\mathcal{D}}<\mathbb{Q},\mathcal{E}_T>,\ L+\sup_{\mathbb{Q}\in\mathcal{D}}<\mathbb{Q},\mathcal{E}_T>\right],\end{aligned}$$

where $L \triangleq \lim_{y\to\infty}\frac{1}{y}\inf_{\mathbb{Q}\in\mathcal{D}}\mathbb{E}\left[\int_0^T V(t,yY_t^{\mathbb{Q}})\,\mu(dt)\right]$. From the Definition 3.1 of the utility function we read $\partial_2 V(t,y)\le -(K_1)^{-1}(y)\to 0$ when $y\to\infty$, so for an $\varepsilon>0$ we can find a constant $C(\varepsilon)$ such that $-V(t,y)\le C(\varepsilon)+\varepsilon y$ for all $t\in[0,T]$ and all $y>0$. To finish the proof, we denote by $\mathfrak{V}_0(\cdot)$ the (strictly convex, decreasing) value function of the dual optimization problem (3.3) when $\mathcal{E}_T\equiv 0$. Then the decrease of $\mathfrak{V}_0(\cdot)$ and L'Hôpital's rule imply

$$\begin{aligned}0 &\le -\mathfrak{V}_0'(\infty) = \lim_{y\to\infty}\frac{-\mathfrak{V}_0(y)}{y} = \lim_{y\to\infty}\sup_{\mathbb{Q}\in\mathcal{D}}\frac{1}{y}\mathbb{E}\left[\int_0^T -V(t,yY_t^{\mathbb{Q}})\,\mu(dt)\right] = -L\\ &\le \lim_{y\to\infty}\sup_{\mathbb{Q}\in\mathcal{D}}\frac{1}{y}\mathbb{E}\left[\int_0^T (C(\varepsilon)+\varepsilon yY_t^{\mathbb{Q}})\,\mu(dt)\right] \le \lim_{y\to\infty}\mathbb{E}\int_0^T(\frac{C(\varepsilon)}{y}+\varepsilon)\,\mu(dt)=\varepsilon.\end{aligned}$$

Consequently, $L=0$, and the claim follows. □

A.4. **Proof of the Main Theorem 3.10.** In this subsection we combine the preceding lemmas and propositions, to complete the proof of Theorem 3.10.

(i) By the concavity of $U(t,\cdot)$ and the Standing Assumption 3.9, we deduce that $\mathfrak{U}(x)<\infty$ for any $x>0$. For $x>0$ we define $c(t)\triangleq x,\ \forall\, t\in[0,T]$. Then $c\in\mathcal{A}^\mu(x+\mathcal{E})$, because the constant consumption-rate process $c(\cdot)\equiv x$ can be financed by the trivial portfolio $H\equiv 0$ and initial wealth only. Since

$$\mathfrak{U}(x)\ \ge\ \mathbb{E}\left[\int_0^T U(t,c(t))\,\mu(dt)\right]=\mathbb{E}\left[\int_0^T U(t,x)\,\mu(dt)\right]\ \ge\ \mathbb{E}\left[\int_0^T \underline{U}(x)\,\mu(dt)\right]=\underline{U}(x)\ >\ -\infty,$$

we conclude that $|\mathfrak{U}(x)|<\infty$ for all $x>0$. The assertion that $|\mathfrak{V}(y)|<\infty$ for all $y>0$ is the content of Lemma A.4.

(ii) $\mathfrak{V}(\cdot)$ is continuously differentiable by Proposition A.6. From the conjugacy relation in Lemma A.3 and the properties of convex conjugation (see Theorem 26.5 in [Roc70]), we deduce the continuous differentiability of $\mathfrak{U}(\cdot)$.

(iii) Follows from Lemma A.3 and the properties of convex conjugation (see Theorem 12.2. in [Roc70]).

(iv) The assertion is a direct consequence of Lemma A.6 and the properties of convex conjugation (see Theorem 26.5. in [Roc70] ).

(vi) Follows from Lemma A.6.

(v) The dual problem has an essentially unique solution $\hat{\mathbb{Q}}^y\in\mathcal{D}$ for any $y>0$, by Proposition A.2 and Remark 10. To establish the result for the primal problem, we pick $x>0$, a solution $\hat{\mathbb{Q}}^y$ of the dual problem corresponding to $y=\mathfrak{U}'(x)$ and define $\hat{c}^x(t)\triangleq I(t,yY_t^{\hat{\mathbb{Q}}^y})$, for all

$t \in [0,T]$. Then the relation $-\mathfrak{V}'(y) = (\mathfrak{U}'(\cdot))^{-1}(y)$, $y > 0$ (see [Roc70], Theorem 26.6) and Proposition A.6 give

$$\mathbb{E}\left[\int_0^T \hat{c}^x(t) Y_t^{\hat{\mathbb{Q}}^y}\, \mu(dt)\right] = -\mathfrak{V}'(y) + \langle \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle = x + \langle \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle,$$

so for any $\mathbb{Q} \in \mathcal{D}$, by Proposition A.5,

$$\mathbb{E}\left[\int_0^T \hat{c}^x(t) Y_t^{\mathbb{Q}}\, \mu(dt)\right] \le \mathbb{E}\left[\int_0^T \hat{c}^x(t) Y_t^{\hat{\mathbb{Q}}^y}\, \mu(dt)\right] + \langle \mathbb{Q}, \mathcal{E}_T\rangle - \langle \hat{\mathbb{Q}}^y, \mathcal{E}_T\rangle = x + \langle \mathbb{Q}, \mathcal{E}_T\rangle.$$

Thus $\hat{c}^x(\cdot) \in \mathcal{A}(x+\mathcal{E})$ by the characterization of admissible consumption processes in Proposition 2.12.

Having established the admissibility of $\hat{c}^x(\cdot)$, we note that

$$\begin{aligned}\mathbb{E}\left[\int_0^T U(t, \hat{c}^x(t))\, \mu(dt)\right] &= \mathbb{E}\left[\int_0^T V(t, yY_t^{\hat{\mathbb{Q}}^y})\, \mu(dt)\right] + \mathbb{E}\left[\int_0^T yY_t^{\hat{\mathbb{Q}}^y} I(t, yY_t^{\hat{\mathbb{Q}}^y})\, \mu(dt)\right] \\ &= \mathfrak{V}(y) - y\mathfrak{V}'(y) = \mathfrak{U}(x),\end{aligned}$$

by the conjugacy relation (iii), the expression of the derivative of the dual value function (v), and the definition of $y$. This closes the duality gap and proves the optimality of $\hat{c}^x(\cdot)$.

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